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A CONVERGENCE ANALYSIS OF AN H-VERSION FINITE ELEMENT METHOD WITH HIGH ORDER ELEMENTS FOR TWO DIMENSIONAL ELASTO-PLASTICITY PROBLEMS

by

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A Convergence Analysis of an H–Version Finite Element Method with High Order Elements for Two Dimensional Elasto–Plasticity Problems

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Abstract

In this paper, we will give an h–version finite element method for a two dimensional nonlinear elasto–plasticity problem. A family of admissible constitutive laws based on the so–called gauge function method is introduced first, and then a high order h–version semi–discretization scheme is presented. The existence and uniqueness of the solution for the semi–discrete problem are guaranteed by using some special properties of the constitutive law, and finally we will show that as the maximum element size $h \longrightarrow 0$, the solution of the semi–discrete problem will converge to the solution of the continuous problem. The high order h–version discretization scheme introduced here is unusual. If the partition of the spatial space only has rectangles or parallelograms involved, then there would not be any limit on the element degree. However, if the partition of the spatial space has some triangular elements, then only certain combinations of finite element spaces for displacement and stress functions can be used. The discretization scheme also provides useful idea for applications of hp–version or high order h–version finite element methods for two dimensional problems where the elasto–plastic body is not a polygon, such as a disk or an annulus.

1 Introduction

Several papers about finite element methods with some theoretical convergence analysis results for two dimensional elasto–plasticity problems have been published in the engineering literature, such as the work of Bonnetier [3], Johnson [10] [11], and Miyoshi [14]. However, in all these approaches, the shape of the domain is assumed to be polygonal and is partitioned into several triangles. The displacement functions are approximated by piecewise linear functions, and the stress functions are approximated by piecewise constant functions. This choice of finite element spaces for the displacement and stress functions makes it easy to use the constitutive law, because both the strain and stress functions will be piecewise constant, and each element can therefore be defined as in either its elastic or plastic state. However, for smooth solutions or over sub–regions where the solution is smooth, higher order polynomial spaces are usually preferred. Moreover, if the domain under consideration has a special shape, such as a rectangular domain or an L shaped domain, then we can just use rectangular elements in the partition. For rectangular elements, we have to apply piecewise bilinear approximation for the displacement function in order to obtain a conforming finite element method. In this case, the corresponding strain functions are piecewise linear instead of piecewise constant. Hence the stress function can no longer be approximated by piecewise constant functions, and we can no longer define a whole element as elastic or plastic. As we will see, the design of a scheme to choose an appropriate constitutive law on high order elements is an important and subtle problem.

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A family of admissible constitutive laws based on the so-called gauge function method was first introduced by Bonnetier [3]. An h-version method with piecewise linear displacement function for problems based on a constitutive law of this kind has been presented in his work. This paper presents a high order h-version finite element method for elasto-plasticity problem based on an admissible constitutive law. An unusual method to define the constitutive law over each element is proposed. The convergence analysis shows that the limit function is guaranteed to satisfy the constitutive law over the whole domain. Two types of elements, namely rectangular and triangular elements, are discussed. In addition, some possible generalizations of our method are also mentioned.

2 A Family of Admissible Constitutive Laws for Two Dimensional Problems

The family of constitutive laws that will be introduced below is based on the following two basic assumptions:

1) Existence of a convex yield surface.
2) The normality condition: the plastic increment is proportional to the outward normal to the yield surface during plastic flow.

There are two main reasons why we choose this family of constitutive laws. First, they are actually a generalization of some of the most commonly used engineering formulations. Second, Bonnetier has shown that the continuous problem based on this family of constitutive laws will be well posed.

For two dimensional problems, we will describe a yield surface by the stress tensor $\sigma = (\sigma_{11}, \sigma_{22}, \sigma_{12})^T$, and a set of hardening parameters (sometimes also called internal parameters):

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)^T \subset \mathcal{U} \subset \mathbf{R}^m$$

where $\mathcal{U}$ is a convex set in $\mathbf{R}^m$.

The elastic set is assumed to be convex, and hence we can think of a function $F(\sigma, \alpha)$ which is a convex function of σ and α as the gauge function of this convex set. More precisely we assume that there exists a function $F: \mathbf{R}^3 \times \mathcal{U} \longmapsto \mathbf{R}$ such that

$A1$). $F(\sigma, \alpha)$ is a convex, and piecewise analytic function of σ, α. (2.1)

$A2$). $F(0,0) = 0$. (2.2)

$A3$). There are constants $\gamma, \Gamma > 0$ such that $\gamma < |\frac{\partial F}{\partial \sigma}|, |\frac{\partial F}{\partial \alpha}| < \Gamma$ (2.3)
uniformly on the set $\{(\sigma, \alpha) \mid F(\sigma, \alpha) = Z_0\}$ for some
$Z_0 > 0$.

where $\frac{\partial F}{\partial \sigma} = \left(\frac{\partial F}{\partial \sigma_{11}}, \frac{\partial F}{\partial \sigma_{22}}, \frac{\partial F}{\partial \sigma_{12}}\right)^T$ and $\frac{\partial F}{\partial \alpha} = \left(\frac{\partial F}{\partial \alpha_1}, \frac{\partial F}{\partial \alpha_2};, \ldots, \frac{\partial F}{\partial \alpha_m}\right)^T$.

The corresponding constitutive law can then be derived by using the two postulates mentioned earlier. Let (σ, α) lie inside the yield surface, i.e., $F(\sigma, \alpha) < Z_0$. Then the material is assumed to be elastic. Therefore the stress and strain rates must satisfy Hooke's law, and there is no change in the hardening parameters. So

$$\begin{cases} \dot{\sigma} = D\dot{\epsilon} \\ \dot{\alpha} = 0 \end{cases} \tag{2.4}$$

where $\dot{\sigma} = \dfrac{\partial \sigma}{\partial t}$, $\dot{\epsilon} = \dfrac{\partial \epsilon}{\partial t}$ etc, and D is the usual elasticity tensor. In plane stress problems, for instance, the elasticity tensor has the form:

$$D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

where E and ν are the Young's modulus and the Poisson's ratio of the material.

If the point (σ, α) lies on the yield surface when $t = t_0$, i.e. $F(\sigma, \alpha)(t_0) = Z_0$, and will move towards the inside of the yield surface afterwards, then we still say that the material is elastic at $t = t_0$. So the stress, strain and hardening parameter rates will still satisfy the equation (2.4).

If the point (σ, α) lies on the yield surface when $t = t_0$, and will remain on the yield surface for $t_0 \le t \le t_0 + \delta$, i.e. $F(\sigma(t), \alpha(t)) = Z_0$, $\forall t \in (t_0, t_0 + \delta)$, then we say that the material is plastic at $t = t_0$ and we will split the strain rate $\dot{\epsilon}$ into elastic and plastic parts

$$\dot{\epsilon} = \dot{\epsilon}^E + \dot{\epsilon}^P, \tag{2.5}$$

and assume that the elastic part is still related to the stress rate by the Hooke's law

$$\dot{\sigma} = D\dot{\epsilon}^E = D(\dot{\epsilon} - \dot{\epsilon}^P). \tag{2.6}$$

Then by the normality condition $\exists \lambda > 0$ such that

$$\begin{bmatrix} \dot{\epsilon}^P \\ -\dot{\alpha} \end{bmatrix} = \lambda \begin{bmatrix} \dfrac{\partial F}{\partial \sigma} \\ \dfrac{\partial F}{\partial \alpha} \end{bmatrix}. \tag{2.7}$$

During the plastic flow, $(\sigma(t), \alpha(t))$ still remain on the yield surface, therefore differentiation of the equation $F(\sigma(t), \alpha(t)) = Z_0$ with respect to time t yields

$$\frac{\partial F}{\partial \sigma}^T \dot{\sigma} + \frac{\partial F}{\partial \alpha}^T \dot{\alpha} = 0. \tag{2.8}$$

Using (2.6), (2.7) and (2.8), we can eliminate $\dot{\epsilon}^P$ to obtain:

$$\left\{\begin{aligned} \dot{\sigma} &= \left[D - \frac{D\dfrac{\partial F}{\partial \sigma}\dfrac{\partial F^T}{\partial \sigma} D}{\dfrac{\partial F^T}{\partial \alpha}\dfrac{\partial F}{\partial \alpha} + \dfrac{\partial F^T}{\partial \sigma} D \dfrac{\partial F}{\partial \sigma}} \right] \dot{\epsilon}, \\ \dot{\alpha} &= -\frac{\dfrac{\partial F^T}{\partial \sigma}\dot{\sigma}}{\dfrac{\partial F^T}{\partial \alpha}\dfrac{\partial F}{\partial \alpha}} \frac{\partial F}{\partial \alpha}. \end{aligned}\right. \tag{2.9}$$

Hence, the constitutive equations based on the gauge function $F(\sigma,\alpha)$ read

$$\left\{\begin{aligned} \dot{\sigma} &= D\dot{\epsilon} \\ \dot{\alpha} &= 0 \end{aligned}\right. \qquad \text{for} \quad (\sigma,\alpha) \in \mathcal{E} \tag{2.10}$$

$$\left\{\begin{aligned} \dot{\sigma} &= \left[D - \frac{D\,\partial_\sigma F\,\partial_\sigma F^T D}{\partial_\alpha F^T \partial_\alpha F + \partial_\sigma F^T D\,\partial_\sigma F} \right] \dot{\epsilon} \\ \dot{\alpha} &= -\frac{\partial_\sigma F^T \dot{\sigma}}{\partial_\alpha F^T \partial_\alpha F}\,\partial_\alpha F \end{aligned}\right. \qquad \text{for} \quad (\sigma,\alpha) \in \mathcal{P} \tag{2.11}$$

where

$$\mathcal{E} = \left\{ (\sigma,\alpha) \;\middle|\; F(\sigma,\alpha) < Z_0 \quad \text{or} \quad F(\sigma,\alpha) = Z_0 \text{ and } \frac{\partial F^T}{\partial \sigma}\dot{\sigma} \le 0 \right\}, \tag{2.12}$$

and

$$\mathcal{P} = \left\{ (\sigma,\alpha) \;\middle|\; F(\sigma,\alpha) = Z_0 \quad \text{and} \quad \frac{\partial F^T}{\partial \sigma}\dot{\sigma} > 0 \right\}. \tag{2.13}$$

The three assumptions A1)~A3) on the gauge function $F(\sigma,\alpha)$ are essential for the proof of the existence and uniqueness of the solution. Actually from the assumption A3), we can get the following:

Proposition 2.1 *Let $\dot{\sigma} = \mathcal{A}(\sigma,\alpha,\dot{\epsilon})$, where $\mathcal{A}$ is the constitutive operator defined by (2.10) and (2.11). Assume that the gauge function $F(\sigma,\alpha)$ satisfies the assumption A3). Then there exist constants $0 < \gamma' < \Gamma'$ independent of the values of σ,α and $\dot{\epsilon}$ such that for any $(\sigma,\alpha) \in \mathbb{R}^3 \times \mathcal{U}$ and any $\dot{\epsilon} \in \mathbb{R}^3$,*

$$\gamma' |\dot{\epsilon}|^2 \le \mathcal{A}(\sigma,\alpha,\dot{\epsilon})^T \dot{\epsilon} \le \Gamma' |\dot{\epsilon}|^2. \tag{2.14}$$

PROOF: Any vector $\dot{\epsilon} = (\dot{\epsilon}_{11}, \dot{\epsilon}_{22}, \dot{\epsilon}_{12})^T$, can be decomposed into a sum of two vectors, one parallel to $\partial_\sigma F$, and the other orthogonal to $\partial_\sigma F$, with respect to the scalar product $\langle a, b \rangle = (Da, b) = a^T D b$, i.e.

$$\dot{\epsilon} = \dot{\epsilon}^{\perp} + \dot{\epsilon}^{\parallel} \tag{2.15}$$

with

$$\begin{aligned} \dot{\epsilon}^{\parallel} &= < \partial_\sigma F, \partial_\sigma F >^{-1} < \partial_\sigma F, \dot{\epsilon} > \partial_\sigma F \\ &= (\partial_\sigma F^T D \partial_\sigma F)^{-1} (\partial_\sigma F^T D \dot{\epsilon}) \partial_\sigma F \end{aligned} \tag{2.16}$$

and

$$(D\dot{\epsilon}^{\perp}, \dot{\epsilon}^{\parallel}) = 0. \tag{2.17}$$

Since D is positive definite and independent of the values of σ, α and $\dot{\epsilon}$, it is sufficient to show that there are constants $0 < \gamma' < \Gamma'$ such that

$$\gamma' \langle \dot{\epsilon}, \dot{\epsilon} \rangle \leq \mathcal{A}(\sigma, \alpha, \dot{\epsilon})^T \dot{\epsilon} \leq \Gamma' \langle \dot{\epsilon}, \dot{\epsilon} \rangle. \tag{2.18}$$

Now suppose the material is elastic, then the constitutive operator $\mathcal{A}$ is defined by (2.10), and we get

$$\mathcal{A}(\sigma, \alpha, \dot{\epsilon})^T \dot{\epsilon} = (D\dot{\epsilon}, \dot{\epsilon}) = \langle \dot{\epsilon}, \dot{\epsilon} \rangle. \tag{2.19}$$

Otherwise, if the material is plastic, the constitutive operator $\mathcal{A}$ will be defined by (2.11) and we can let

$$\mu = \left[\partial_\alpha F^T \partial_\alpha F + \partial_\sigma F^T D \partial_\sigma \right]^{-1},$$

and get

$$\begin{aligned} \mathcal{A}(\sigma, \alpha, \dot{\epsilon})^T \dot{\epsilon} &= \left[(D - \mu D \partial_\sigma F \partial_\sigma F^T D) \dot{\epsilon}, \dot{\epsilon} \right] \\ &= \langle \dot{\epsilon} - \mu (\partial_\sigma F^T D \dot{\epsilon}) \partial_\sigma F, \dot{\epsilon} \rangle \\ &= \langle \dot{\epsilon}^{\perp}, \dot{\epsilon}^{\perp} \rangle + \langle \dot{\epsilon}^{\parallel} - \mu (\partial_\sigma F^T D \dot{\epsilon}) \partial_\sigma F, \dot{\epsilon}^{\parallel} \rangle. \end{aligned}$$

From the definition of $\dot{\epsilon}^{\parallel}$, we have

$$\begin{aligned} \mathcal{A}(\sigma, \alpha, \dot{\epsilon})^T \dot{\epsilon} &= \langle \dot{\epsilon}^{\perp}, \dot{\epsilon}^{\perp} \rangle + (1 - \mu (\partial_\sigma F^T D \partial_\sigma F)) \langle \dot{\epsilon}^{\parallel}, \dot{\epsilon}^{\parallel} \rangle \\ &= \langle \dot{\epsilon}^{\perp}, \dot{\epsilon}^{\perp} \rangle + \mu (\partial_\alpha F^T \partial_\alpha F) \langle \dot{\epsilon}^{\parallel}, \dot{\epsilon}^{\parallel} \rangle. \end{aligned} \tag{2.20}$$

By virtue of the assumption A3), the term $\mu \partial_\alpha F^T \partial_\alpha F$ is bounded from below and above:

$$\frac{\gamma^2}{\Gamma^2(1 + \|D\|)} \leq \mu (\partial_\alpha F^T \partial_\alpha F) \leq 1$$

which shows that (2.14) holds for $\gamma' = \dfrac{\gamma^2}{\Gamma^2(1 + \|D\|)}$ and $\Gamma' = 1$. □

Definition 2.1 *The family of the constitutive laws (2.10)~ (2.13) based on the gauge function of the yield surface, $F(\sigma, \alpha)$ which satisfies (2.1)~ (2.3) will be called admissible constitutive laws or briefly admissible laws.*

As we mentioned earlier, the admissible constitutive laws are actually a generalization of some of the most commonly adopted engineering formulations. For instance, the kinematic hardening law for two dimensional problem (also called Ziegler's rule), can be formulated as follows:

Let $\beta = (\beta_{11}, \beta_{22}, \beta_{12})^T$ be the hardening parameters, and $\eta > 0$ be the constant coefficient of hardening rate. The Von Mises surface is chosen to be the yield surface, and it has the form:

$$f(\sigma - \beta) = Z_0 \tag{2.21}$$

with $Z_0 > 0$ a constant and

$$f(\sigma) = \left(\sigma_{11}^2 + \sigma_{22}^2 - \sigma_{11}\sigma_{22} + 3\sigma_{12}^2\right)^{1/2}.$$

The constitutive equations based on Ziegler's rule then take the form:

$$\begin{cases} \dot{\sigma} = D\dot{\epsilon} & \text{if} \quad f(\sigma - \beta) < Z_0 \\ \dot{\beta} = 0 & \text{or} \quad f(\sigma - \beta) = Z_0 \quad \text{and} \quad \partial f^T \dot{\sigma} \leq 0 \end{cases} \tag{2.22}$$

$$\begin{cases} \dot{\sigma} = \left[D - (\eta + \partial f^T D \partial f)^{-1} (D \partial f \partial f^T D) \right] \dot{\epsilon} \\ \dot{\beta} = (\sigma - \beta) \dfrac{\partial f^T \dot{\sigma}}{Z_0} \end{cases} \quad \text{if} \;\; f(\sigma - \beta) = Z_0 \;\; \text{and} \;\; \partial f^T \dot{\sigma} > 0 \tag{2.23}$$

with the rate of the plastic strain defined as:

$$\dot{\epsilon}^p = \eta^{-1} (\partial f^T \dot{\sigma}) \partial f. \tag{2.24}$$

Now let $\mathbf{R}$ be the matrix

$$\mathbf{R} = \begin{bmatrix} 4/3 & 2/3 & 0 \\ 2/3 & 4/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

and $\sqrt{\mathbf{R}}$ be one of its square roots. We can then define another set of hardening parameters $\alpha = (\sqrt{\eta}\sqrt{\mathbf{R}})^{-1}\beta$, and a gauge function of the form:

$$F(\sigma, \alpha) = f(\sigma - \sqrt{\eta}\sqrt{\mathbf{R}}\,\alpha). \tag{2.25}$$

Obviously the gauge function $F(\sigma, \alpha)$ defined by (2.25) is a convex and analytic function of σ and α, and satisfies the equation $F(0,0) = 0$. A simple computation also shows that on the set $\{ (\sigma, \alpha) \in \mathbf{R}^3 \times \mathbf{R}^3 \mid F(\sigma, \alpha) = Z_0 \}$, the derivative $\partial_\sigma F$ satisfies the inequalities

$$\frac{1}{2} \leq |\partial_\sigma F| \leq \sqrt{2}.$$

Since $\mathbf{R}$ is positive definite, $\partial_\alpha F = -\sqrt{\eta}\sqrt{\mathbf{R}}\,\partial_\sigma F$ is also bounded uniformly on this set. Hence the gauge function F satisfies all three of our assumptions. It is not difficult to check that in (2.10)~(2.13), if we use the gauge function F defined as in (2.25), we will then get a set of constitutive equations which is equivalent to the equations (2.22) and (2.23) after the variable substitution $\alpha = (\sqrt{\eta}\sqrt{\mathbf{R}})^{-1}\beta$. Hence we can consider the kinematic hardening law as a particular case of our general admissible constitutive laws.

Similarly the two dimensional pure isotropic hardening law can be cast into our admissible form by using a gauge function of the form:

$$F(\sigma,\alpha) = f(\sigma) - h(\alpha),$$

where $\alpha \in \mathcal{U} \subset \mathbb{R}$ is the isotropic hardening parameter and $h(\alpha)$ is a concave monotonically increasing function of the variable α with $h(0) = 0$.

3 Two Dimensional Quasi–static Elasto–Plastic Problems with Constitutive Law Based on A Gauge Function Method

A typical two dimensional quasi–static elasto–plastic problem with the constitutive law based on the gauge function method can be formulated as follows:

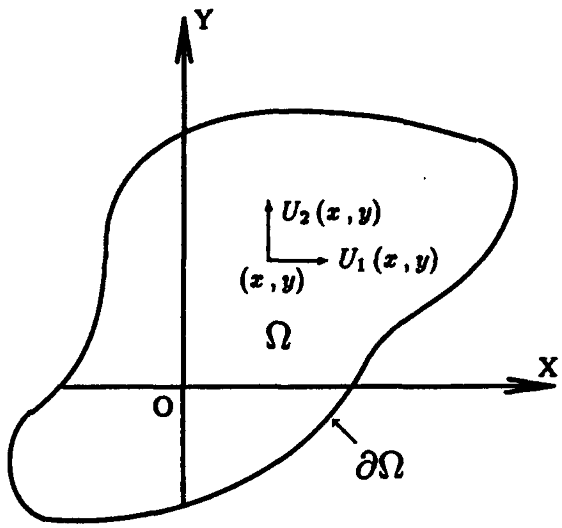

FIGURE 3.1 A bounded domain in $\mathbb{R}^2$

Let Ω, a bounded domain in $\mathbb{R}^2$ (cf. Figure 3.1), be the reference configuration of an elasto–plastic body. We denote by $U = (U_1, U_2)$ the displacement field of the body where $U_1(x,y)$, $U_2(x,y)$ are the displacements at the point (x,y) in the x and y directions respectively. Let $\sigma = (\sigma_{11}, \sigma_{22}, \sigma_{12})^T$ be the stress tensor, where σ_{11} and σ_{22} are the normal stress in the x and y directions, and σ_{12} is the shear stress. We will use $\epsilon = (\epsilon_{11}, \epsilon_{22}, \epsilon_{12})^T$ to denote the strain tensor, where the normal strains ϵ_{11} and ϵ_{22} and the shear strain ϵ_{12} are defined in the normal engineering formula:

$$\epsilon_{11} = \frac{\partial U_1}{\partial x} \quad ; \quad \epsilon_{22} = \frac{\partial U_2}{\partial y} \quad ; \quad \epsilon_{12} = \frac{\partial U_1}{\partial y} + \frac{\partial U_2}{\partial x} \quad . \tag{3.1}$$

Finally $I = [0, T]$ will be the time interval of interest, and $f = (f_1, f_2)^T$ is the body traction acting on the elasto–plastic body Ω. Then a general two dimensional quasi–static elasto–plastic problem with the constitutive law based on the gauge function method is as follows.

To find the displacement functions $U(x,y,t)$ and the stress functions $\sigma(x,y,t)$, such that $\sigma(x,y,t)$ satisfies the following equilibrium equations:

$$\begin{cases} -\dfrac{\partial \sigma_{11}}{\partial x}(x,y,t) - \dfrac{\partial \sigma_{12}}{\partial y}(x,y,t) = f_1(x,y,t) \\ -\dfrac{\partial \sigma_{21}}{\partial x}(x,y,t) - \dfrac{\partial \sigma_{22}}{\partial y}(x,y,t) = f_2(x,y,t) \end{cases} \quad a.e. \text{ on } \Omega \times I \tag{3.2}$$

with the boundary condition:

$$U_1(x,y,t) = U_2(x,y,t) = 0 \qquad \text{for} \quad (x,y) \in \partial\Omega, \tag{3.3}$$

and the stress function $\sigma(x,y,t)$ defined by the following constitutive equations:

$$\begin{cases} \dot\sigma(x,y,t) = D\dot\epsilon(x,y,t) \\ \dot\alpha(x,y,t) = 0 \end{cases} \qquad \text{for} \quad (\sigma,\alpha) \in \mathcal{E} \tag{3.4}$$

$$\begin{cases} \dot\sigma(x,y,t) = (D - D')\dot\epsilon(x,y,t) \\ \dot\alpha(x,y,t) = \dfrac{(\partial_\sigma F^T\dot\sigma)(x,y,t)}{(\partial_\sigma F^T\partial_\sigma F)(x,y,t)}\partial_\alpha F(x,y,t) \end{cases} \qquad \text{for} \quad (\sigma,\alpha) \in \mathcal{P} \tag{3.5}$$

where D' is defined as

$$D'(x,y,t) = \frac{(D\partial_\sigma F\partial_\sigma F^T D)(x,y,t)}{(\partial_\alpha F^T\partial_\alpha F + \partial_\sigma F^T D\,\partial_\sigma F)(x,y,t)}. \tag{3.6}$$

The stress, strain and hardening parameter functions satisfy the following initial conditions:

$$\begin{array}{lll} \sigma_{ij}(x,y,0) = 0 \;\; ; \;\; U_i(x,y,0) = 0 & & i,j = 1,2 \;, \\ \alpha_i(x,y,0) = 0 & i = 1,2,\ldots,m \;. & \end{array} \tag{3.7}$$

Of course here we have to require that the body traction functions satisfy the compatibility conditions:

$$f_i(x,y,0) = 0, \qquad i = 1,2 \;. \tag{3.8}$$

Finally $\mathcal{E}$, $\mathcal{P}$ are the elastic and the plastic sets defined by:

$$\mathcal{E} = \left\{ (\sigma,\alpha) \in \mathbb{R}^3 \times \mathbb{R}^m \;\middle|\; \text{if} \;\; \begin{array}{ll} F(\sigma,\alpha) < Z_0 & \text{or} \\ F(\sigma,\alpha) = Z_0 \;\&\; \partial_\sigma F^T(\sigma,\alpha)\dot\sigma \le 0 & \end{array} \right\} \tag{3.9}$$

$$\mathcal{P} = \left\{ (\sigma,\alpha) \in \mathbb{R}^3 \times \mathbb{R}^m \;\middle|\; \text{if} \;\; \begin{array}{l} F(\sigma,\alpha) = Z_0 \;\; \text{and} \\ \partial_\sigma F^T(\sigma,\alpha)\dot\sigma > 0 \end{array} \right\} \tag{3.10}$$

For the above two dimensional elasto–plasticity problem, Bonnetier [3] has proved the following theorem:

Theorem 3.1 *If the gauge function $F(\sigma,\alpha)$ is admissible, i.e. it satisfies all the three assumptions A1)∼A3), and if the body traction functions $f_i(x,y,t)$, $i = 1,2$ are in $L^2(\Omega)$ for all fixed t, and piecewise analytic in t for (x,y) a.e. on Ω. Then the two dimensional elasto–plasticity problem (3.2) ∼ (3.10) has a unique solution. Moreover, the solution is also piecewise analytic in time t.*

4 Semi-Discrete Approximation of the Quasi-Static Problem

Consider first a special case where $\Omega \subset \mathbb{R}^2$ is a bounded domain which can be partitioned into rectangles $\{T_n\}$ of size h and the aspect ratios satisfy $\alpha_0 < \alpha < \alpha_1$ where α_0 and α_1 are independent of the element size h. Let $\mathcal{T}_h$ be such a partition, $\mathcal{N}$ be the set of nodes of $\Omega \backslash \partial\Omega$, and $\{\phi_n\}$, $n \in \mathcal{N}$ be the piecewise bilinear basis functions such that $\phi_n = 1$ at the node n and $\phi_n = 0$ at other nodes.

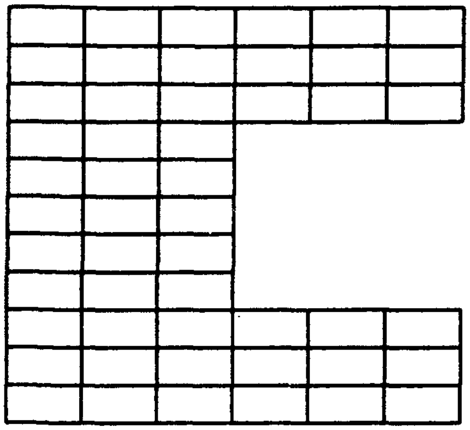

FIGURE 4.1 A domain Ω which can be partitioned into several rectangles

Denote by $Q_{1,0}(\Omega_h)$ the set of continuous piecewise bilinear functions defined over Ω such that they equal zero on the boundary $\partial\Omega$, *i.e.*

$$Q_{1,0}(\Omega_h) = \left\{ h(x,y) \in H_0^1(\Omega) \;\middle|\; \begin{array}{c} h(x,y)|_{T_n} = a\,x + b\,y + c\,x\,y + d \\ h(x,y)|_{\partial\Omega} = 0 \end{array} \right\}.$$

Thus any function $H(x,y)$ in $Q_{1,0}(\Omega_h)$ can be written in the form:

$$H(x,y) = \sum_{n \in \mathcal{N}} H_n \phi_n(x,y).$$

Meanwhile denote by $\tilde{Q}_1(\Omega_h)$ the set of all piecewise bilinear functions defined over Ω, *i.e.*

$$\tilde{Q}_1(\Omega_h) = \left\{ \tilde{h}(x,y) \in L^2(\Omega) \;\middle|\; \tilde{h}(x,y)\Big|_{T_n} = a\,x + b\,y + c\,x\,y + d \right\}.$$

Notice that the functions in $\tilde{Q}_1(\Omega_h)$ are not necessarily continuous over the domain Ω. Let $\tilde{\mathcal{N}}$ be the total degrees of freedom of the finite element space $\tilde{Q}_1(\Omega_h)$, and $\left\{\tilde{\phi}_n\right\}$, $n \in \tilde{\mathcal{N}}$ be the piecewise bilinear basis functions of the space $\tilde{Q}_1(\Omega_h)$. So any function $\tilde{H}(x,y)$ in $\tilde{Q}_1(\Omega_h)$ can be expressed in the form:

$$\tilde{H}(x,y) = \sum_{n \in \tilde{\mathcal{N}}} \tilde{H}_n \tilde{\phi}_n(x,y).$$

For piecewise bilinear functions, we can use four Gaussian points on each rectangular element. Then by using the interpolation method, any function in the space $\tilde{Q}_1(\Omega_h)$ will be uniquely determined by its values at these four Gaussian points.

Now we can formulate our semi-discrete plasticity problem as follows.

To seek the solution (U, σ, α), such that for any fixed t, $U = (U_1, U_2)^T \in (Q_{1,0}(\Omega_h))^2$, $\sigma = (\sigma_{11}, \sigma_{22}, \sigma_{12})^T \in \left(\tilde{Q}_1(\Omega_h)\right)^3$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)^T \in \left(\tilde{Q}_1(\Omega_h)\right)^m$, and they are in the form

$$
\begin{aligned}
U_i &= \sum_{n \in \mathcal{N}} U_{i,n}(t)\phi_n(x) && \text{with} \quad U_{i,n} \in C_+^{0,\infty}(0,T) && i = 1,2 \\
\sigma_{ij} &= \sum_{n \in \tilde{\mathcal{N}}} \sigma_{ij,n}(t)\tilde{\phi}_n(x) && \text{with} \quad \sigma_{ij,n} \in C_+^{0,\infty}(0,T) && i,j = 1,2 \\
\alpha_i &= \sum_{n \in \tilde{\mathcal{N}}} \alpha_{i,n}(t)\tilde{\phi}_n(x) && \text{with} \quad \alpha_{i,n} \in C_+^{0,\infty}(0,T) && i = 1,\dots,m
\end{aligned}
$$

where $C_+^{0,\infty}(0,T)$ is the set of functions, continuous on $[0,T]$, and have a uniformly bounded right derivative of any order at each point on $[0,T)$. The stress functions satisfy the equilibrium equations:

$$
\left\{
\begin{aligned}
\int_\Omega (\sigma_{11}, \frac{\partial \phi_n}{\partial x_1}) + (\sigma_{12}, \frac{\partial \phi_n}{\partial x_2}) &= \int_\Omega (f_1, \phi_n) \\
\int_\Omega (\sigma_{21}, \frac{\partial \phi_n}{\partial x_1}) + (\sigma_{22}, \frac{\partial \phi_n}{\partial x_2}) &= \int_\Omega (f_2, \phi_n)
\end{aligned}
\right.
\qquad \forall \; n \in \mathcal{N} \tag{4.1}
$$

Here the body traction $f = (f_1, f_2)$ is piecewise analytic in time and for each $t \in [0,T]$, $f(\cdot,t) \in L^2(\Omega)$. On each element, the corresponding constitutive equations are given by:

$$
\left\{
\begin{aligned}
\dot{\sigma}(\hat{x}_k) &= D\dot{\epsilon}(\hat{x}_k) \\
\dot{\alpha}(\hat{x}_k) &= 0
\end{aligned}
\right.
\qquad if \quad
\left\{
\begin{array}{l}
F(\sigma(\hat{x}_k), \alpha(\hat{x}_k)) < Z_0 \\
\quad \text{or} \\
\partial_\sigma F^T(\hat{x}_k)\dot{\sigma}(\hat{x}_k) \le 0
\end{array}
\right. \tag{4.2}
$$

$$
\left\{
\begin{aligned}
\dot{\sigma}(\hat{x}_k) &= (D - D')(\hat{x}_k)\,\dot{\epsilon}(\hat{x}_k) \\
\dot{\alpha}(\hat{x}_k) &= -\frac{\partial_\sigma F^T(\hat{x}_k)\dot{\sigma}(\hat{x}_k)}{(\partial_\alpha F^T \partial_\alpha F)(\hat{x}_k)}\partial_\alpha F(\hat{x}_k)
\end{aligned}
\right.
\qquad if \quad
\left\{
\begin{array}{l}
F(\sigma(\hat{x}_k), \alpha(\hat{x}_k)) = Z_0 \\
\quad \text{and} \\
\partial_\sigma F^T(\hat{x}_k)\dot{\sigma}(\hat{x}_k) > 0
\end{array}
\right. \tag{4.3}
$$

where $\hat{x}_k$ $(k = 1,2,3,4) \in \hat{G}$ and $\hat{G}$ is the set of Gaussian points of all the rectangles in the domain Ω, and

$$
D'(\hat{x}_k) = \frac{D\partial_\sigma F \partial_\sigma F^T D}{\partial_\alpha F^T \partial_\alpha F + \partial_\sigma F^T D \partial_\sigma F}(\hat{x}_k). \tag{4.4}
$$

The initial conditions are:

$$
U(0,x) = 0 \quad , \quad \sigma(0,x) = 0 \quad , \quad \alpha(0,x) = 0 \quad . \tag{4.5}
$$

Of course here again we assume that the traction functions satisfy the compatibility conditions:

$$
f_1(0,x) = f_2(0,x) = 0 \quad .
$$

To simplify the statement of our proof, in the following context we will assume that the body traction functions are analytic in time, and the gauge function $F(\sigma,\alpha)$ is also analytic in σ and α. (Similar results still hold if f is only piecewise analytic in time, and $F(\sigma,\alpha)$ is only piecewise analytic in σ and α.)

5 Existence and Uniqueness of the Solution of the Semi-discrete Problem

Note that $U = (U_1, U_2)$ is chosen to be piecewise bilinear. From the relation between strain and displacement (3.1), we know that the strain functions as well as the strain rate functions will be piecewise linear. So now instead of the whole element, we can define a Gaussian point $\hat{x}_k \in \hat{G}$ of a rectangular element to be elastic at t_0 (resp. plastic) if it satisfies (4.2) (resp. (4.3)) at the point $\hat{x}_k$. Therefore from their values at the Gaussian points we can define the functions σ, α, $\dot{\sigma}$ and $\dot{\alpha}$ to be some piecewise bilinear functions.

Consequently we can now define the set of elastic Gaussian points and the set of plastic Gaussian points as follows:

$$\hat{\mathcal{E}} = \hat{\mathcal{E}}(\dot{U}) = \left\{ x \in \hat{G} \;\middle|\; \begin{array}{ll} F(\sigma,\alpha)(x) < Z_0 & \text{or} \\ F(\sigma,\alpha)(x) = Z_0 \text{ and } \partial_\sigma F^T D\,\dot{\epsilon}(U)(x) \le 0 \end{array} \right\} \tag{5.1}$$

and

$$\hat{\mathcal{P}} = \hat{\mathcal{P}}(\dot{U}) = \left\{ x \in \hat{G} \mid F(\sigma,\alpha)(x) = Z_0 \text{ and } \partial_\sigma F^T D\,\dot{\epsilon}(U)(x) > 0 \right\}, \tag{5.2}$$

where $\dot{\epsilon}(U) = \dfrac{\partial \epsilon(U)}{\partial t} = \epsilon(\dfrac{\partial U}{\partial t}) = \epsilon(\dot{U})$, and $\hat{G}$ is the set of all the Gaussian points in the domain Ω.

Using the constitutive equations given by (4.2) and (4.3), we can easily verify the following:

Proposition 5.1 *The elastic and the plastic Gaussian point sets, $\hat{\mathcal{E}}$ and $\hat{\mathcal{P}}$, defined by (5.1) and (5.2) can also be expressed in terms of the stress tensor:* $\forall\, U \in (H_0^1(\Omega))^2$,

$$\hat{\mathcal{E}}(\dot{U}) = \left\{ x \in \hat{G} \;\middle|\; \begin{array}{ll} F(\sigma,\alpha)(x) < Z_0 & \textit{or} \\ F(\sigma,\alpha)(x) = Z_0 & \&\;\; \partial_\sigma F^T \dot{\sigma}(U)(x) \le 0 \end{array} \right\} \tag{5.3}$$

$$\hat{\mathcal{P}}(\dot{U}) = \left\{ x \in \hat{G} \mid F(\sigma,\alpha)(x) = Z_0 \;\; \textit{and} \;\; \partial_\sigma F^T \dot{\sigma}(U)(x) > 0 \right\} \tag{5.4}$$

where the stress tensor $\dot{\sigma}(U)(x)$ is defined by the constitutive equations (4.2) and (4.3).

Let $(\cdot,\cdot)$ be the standard vector inner product defined on $\mathbb{R}^3$. For any given displacement function $U(x,t)$ and any Gaussian point $\hat{x}$, by using (3.1) we can easily find the values of the corresponding strain rate at that point, $\dot{\epsilon}(\hat{x},t)$. Then by using the constitutive equations (4.2) and (4.3) we can find the values of the corresponding stress rate at that point, $\dot{\sigma}(\hat{x},t)$. Hence at each Gaussian point $\hat{x}$, we can defined a functional $G(U(\hat{x}))$ such that:

$$G(U(\hat{x})) = (\dot{\sigma}(U(\hat{x})), \dot{\epsilon}(U(\hat{x}))).$$

For this functional, we can give the following Lemma which is important in the proof of existence and uniqueness of the solution for the semi-discrete problem.

Lemma 5.1 *Let* $(\dot\sigma(\hat x_k), \dot\alpha(\hat x_k)) \in \mathbb{R}^3 \times \mathcal{U}$ *be determined by equation (4.2) and (4.3). Then the functional* $G(U(\hat x_k)) = (\dot\sigma(U(\hat x_k)), \dot\epsilon(U(\hat x_k)))$ *, defined on the set* $(Q_{1,0}(T_n))^2$ *, is strictly convex. Here* T_n *is any rectangular element of the partition* $\mathcal{T}_h$ *.*

Proof: Let $U, V \in (Q_{1,0}(T_n))^2$, $\theta \in (0,1)$ and $W = \theta U + (1-\theta)V$. We define:

$$\mathcal{E}_0 = \left\{ \hat x_k \in T_n \cap \widehat G \mid F(\sigma, \alpha) < Z_0 \right\}$$
$$\mathcal{P}_1 = \left\{ \hat x_k \in T_n \cap \widehat G \mid F(\sigma, \alpha) = Z_0 ,\ \partial_\sigma F^T D \dot\epsilon(U) > 0 ,\ \partial_\sigma F^T D \dot\epsilon(V) > 0 \right\}$$
$$\mathcal{P}_2 = \left\{ \hat x_k \in T_n \cap \widehat G \mid F(\sigma, \alpha) = Z_0 ,\ \partial_\sigma F^T D \dot\epsilon(U) \le 0 ,\ \partial_\sigma F^T D \dot\epsilon(V) \le 0 \right\}$$
$$\mathcal{P}_3 = \left\{ \hat x_k \in T_n \cap \widehat G \mid F(\sigma, \alpha) = Z_0 ,\ \partial_\sigma F^T D \dot\epsilon(U) > 0 ,\ \partial_\sigma F^T D \dot\epsilon(V) \le 0 \right\}$$
$$\mathcal{P}_4 = \left\{ \hat x_k \in T_n \cap \widehat G \mid F(\sigma, \alpha) = Z_0 ,\ \partial_\sigma F^T D \dot\epsilon(U) \le 0 ,\ \partial_\sigma F^T D \dot\epsilon(V) > 0 \right\}$$

1). On $\mathcal{E}_0 \bigcup \mathcal{P}_2$, all three functions, U, V and W, are corresponding to the elastic state. So we get

$$\begin{aligned} G(W) &= (\dot\sigma(W), \dot\epsilon(W)) = (D\dot\epsilon(W), \dot\epsilon(W)) \\ &< \theta(D\dot\epsilon(U), \dot\epsilon(U)) + (1-\theta)(D\dot\epsilon(V), \dot\epsilon(V)) \\ &= \theta(\dot\sigma(U), \dot\epsilon(U)) + (1-\theta)(\dot\sigma(V), \dot\epsilon(V)) \\ &= \theta G(U) + (1-\theta) G(V) \quad . \end{aligned}$$

2). On $\mathcal{P}_1$, all three functions, U, V and W, are corresponding to the plastic state. Hence we have

$$\begin{aligned} G(W) &= (\dot\sigma(W), \dot\epsilon(W)) = ((D-D')\dot\epsilon(W), \dot\epsilon(W)) \\ &< \theta((D-D')\dot\epsilon(U), \dot\epsilon(U)) + (1-\theta)((D-D')\dot\epsilon(V), \dot\epsilon(V)) \\ &= \theta(\dot\sigma(U), \dot\epsilon(U)) + (1-\theta)(\dot\sigma(V), \dot\epsilon(V)) \\ &= \theta G(U) + (1-\theta) G(V) \quad , \end{aligned}$$

where D' is defined as (4.4) which is independent of the values of $\dot U$ and $\dot V$, and only dependents on the values of σ and α.

3). On $\mathcal{P}_3$, we have the case where U is corresponding to the elastic state, and V corresponding to the plastic state. Decompose the vector $\dot\epsilon = (\dot\epsilon_{11}, \dot\epsilon_{22}, \dot\epsilon_{12})^T$ into a sum of two vectors, one parallel to $\partial_\sigma F(\sigma(\hat x_k), \alpha(\hat x_k))$, and the other orthogonal to $\partial_\sigma F(\sigma(\hat x_k), \alpha(\hat x_k))$ with respect to the scalar product $\langle \cdot, \cdot \rangle$, *i.e.*

$$\dot\epsilon = \dot\epsilon^{\perp} + \dot\epsilon^{\parallel} \tag{5.5}$$

and

$$(D\dot\epsilon^{\perp}, \dot\epsilon^{\parallel}) = 0 \quad . \tag{5.6}$$

Obviously we have

$$\begin{cases} \dot\epsilon^{\perp}(W) = \theta\dot\epsilon^{\perp}(U) + (1-\theta)\dot\epsilon^{\perp}(V) \\ \dot\epsilon^{\parallel}(W) = \theta\dot\epsilon^{\parallel}(U) + (1-\theta)\dot\epsilon^{\parallel}(V) \end{cases} \tag{5.7}$$

Case1. Suppose $\partial_\sigma F^T D\dot{\epsilon}(W) > 0$, then

$$\begin{aligned}(\dot{\sigma}(W), \dot{\epsilon}(W)) &= ((D - D')\dot{\epsilon}(W), \dot{\epsilon}(W)) \\ &= \langle \dot{\epsilon}(W) - \mu\partial_\sigma F^T D\dot{\epsilon}(W)\partial_\sigma F, \dot{\epsilon}(W)\rangle \\ &= \langle \dot{\epsilon}^\perp(W), \dot{\epsilon}^\perp\rangle \\ &\quad + \langle \dot{\epsilon}^\parallel(W) - \mu\partial_\sigma F^T D\dot{\epsilon}(W)\partial_\sigma F, \dot{\epsilon}^\parallel(W)\rangle ,\end{aligned}$$

where $\mu = (\partial_\alpha F^T \partial_\alpha F + \partial_\sigma F^T D\partial_\sigma F)(\sigma(\hat{x}_k), \alpha(\hat{x}_k))$. Hence by the definition of $\dot{\epsilon}^\parallel$, we have

$$(\dot{\sigma}(W), \dot{\epsilon}(W)) = \langle \dot{\epsilon}^\perp(W), \dot{\epsilon}^\perp(W)\rangle + K(\partial_\sigma F^T D\dot{\epsilon}(W))^2 , \tag{5.8}$$

where

$$K = \left[\frac{1}{\partial_\sigma F^T D\partial_\sigma F} - \frac{1}{\partial_\sigma F^T D\partial_\sigma F + \partial_\alpha F^T \partial_\alpha F}\right]$$

is a positive number independent of the values of $\dot{U}$ and $\dot{V}$. From the fact that

$$\begin{cases} \partial_\sigma F^T D\dot{\epsilon}(W) = \theta\partial_\sigma F^T D\dot{\epsilon}(U) + (1-\theta)\partial_\sigma F^T D\dot{\epsilon}(V) > 0 \\ \partial_\sigma F^T D\dot{\epsilon}(V) \le 0 \end{cases}$$

we get

$$0 < \partial_\sigma F^T D\dot{\epsilon}(W) \le \theta\partial_\sigma F^T D\dot{\epsilon}(U) .$$

On the other hand, by the convexity of the scalar product $\langle\cdot,\cdot\rangle$ and the equation (5.7), we have

$$\langle \dot{\epsilon}^\perp(W), \dot{\epsilon}^\perp(W)\rangle < \theta\langle \dot{\epsilon}^\perp(U), \dot{\epsilon}^\perp(U)\rangle + (1-\theta)\langle \dot{\epsilon}^\perp(V), \dot{\epsilon}^\perp(V)\rangle .$$

Hence the equation (5.8) becomes

$$\begin{aligned}(\dot{\sigma}(W), \dot{\epsilon}(W)) < \; &\theta\langle \dot{\epsilon}^\perp(U), \dot{\epsilon}^\perp(U)\rangle + (1-\theta)\langle \dot{\epsilon}^\perp(V), \dot{\epsilon}^\perp(V)\rangle \\ &+ \theta K(\partial_\sigma F^T D\dot{\epsilon}(U))^2 ,\end{aligned}$$

or

$$\begin{aligned}(\dot{\sigma}(W), \dot{\epsilon}(W)) < \theta &\left[\langle \dot{\epsilon}^\perp(U), \dot{\epsilon}^\perp(U)\rangle + K(\partial_\sigma F^T D\dot{\epsilon}(U))^2\right] \\ &+ (1-\theta)\langle \dot{\epsilon}(V), \dot{\epsilon}(V)\rangle .\end{aligned} \tag{5.9}$$

Using (5.8) with W replaced by U (since $\dot{\sigma}(U)$ corresponding to the plastic state), we finally have

$$(\dot{\sigma}(W), \dot{\epsilon}(W)) < \theta(\dot{\sigma}(U), \dot{\epsilon}(U)) + (1-\theta)(\dot{\sigma}(V), \dot{\epsilon}(V)) .$$

Case2. Suppose $\partial_\sigma F^T D\dot{\epsilon}(W) \le 0$, then

$$\begin{aligned}(\dot{\sigma}(W), \dot{\epsilon}(W)) &= \langle \dot{\epsilon}(W), \dot{\epsilon}(W)\rangle \\ &= \langle \dot{\epsilon}^\perp(W), \dot{\epsilon}^\perp(W)\rangle + \langle \dot{\epsilon}^\parallel(W), \dot{\epsilon}^\parallel(W)\rangle ,\end{aligned}$$

or

$$(\dot\sigma(W), \dot\epsilon(W)) = \langle \dot\epsilon^{\perp}(W), \dot\epsilon^{\perp}(W)\rangle + \frac{(\partial_\sigma F^T D\dot\epsilon(W))^2}{\langle \partial_\sigma F, \partial_\sigma F\rangle} \quad . \tag{5.10}$$

From the fact that

$$\begin{cases} \partial_\sigma F^T D\dot\epsilon(W) &= \theta\,\partial_\sigma F^T D\dot\epsilon(U) + (1-\theta)\,\partial_\sigma F^T D\dot\epsilon(V) \\ \partial_\sigma F^T D\dot\epsilon(U) &> 0 \end{cases} ,$$

we get

$$(1-\theta)\,\partial_\sigma F^T D\dot\epsilon(V) \le \partial_\sigma F^T D\dot\epsilon(W) \quad .$$

Hence (5.10) can be rewritten as

$$\begin{aligned}(\dot\sigma(W), \dot\epsilon(W)) \;<\; & (1-\theta)\left[\langle \dot\epsilon^{\perp}(V), \dot\epsilon^{\perp}(V)\rangle + \frac{(\partial_\sigma F^T D\dot\epsilon(V))^2}{\langle \partial_\sigma F, \partial_\sigma F\rangle}\right] \\ & +\theta\langle \dot\epsilon^{\perp}(U), \dot\epsilon^{\perp}(U)\rangle \quad . \end{aligned}$$

Replacing W by U in (5.8), we get

$$\langle \dot\epsilon^{\perp}(U), \dot\epsilon^{\perp}(U)\rangle \le (\dot\sigma(U), \dot\epsilon(U)) \quad .$$

Replacing W by V in (5.10) (since $\dot\sigma(V)$ is corresponding to the elastic state), we get

$$(\dot\sigma(V), \dot\epsilon(V)) = \langle \dot\epsilon^{\perp}(V), \dot\epsilon^{\perp}(V)\rangle + \frac{(\partial_\sigma F^T D\dot\epsilon(V))^2}{\langle \partial_\sigma F, \partial_\sigma F\rangle} \quad .$$

So again we get

$$(\dot\sigma(W), \dot\epsilon(W)) < (1-\theta)(\dot\sigma(V), \dot\epsilon(V)) + \theta(\dot\sigma(U), \dot\epsilon(U)) \quad .$$

4). On $\mathcal{P}_4$, we can prove the same result by changing the roles of U and V in 3).

From the above discussion we see that the functional $G(U(\hat{x}_k))$ is strictly convex, and this completes our proof of the lemma. □

Now let $\dot f = (\dot f_1, \dot f_2) \in (L^2(\Omega))^2$ be the incremental body traction, we can define a functional as follows:

For any

$$\dot U = \begin{bmatrix} \dot U_1 \\ \dot U_2 \end{bmatrix} \in (H_0^1(\Omega))^2 \quad ,$$

let

$$\mathcal{F}(\sigma, \alpha, \dot U) = \frac{1}{2}\int_\Omega (\dot\sigma(U), \dot\epsilon(U))\,dx - \int_\Omega (\dot f, \dot U)\,dx \tag{5.11}$$

where $\dot\sigma(U) \in \left(\tilde Q_1(\Omega_h)\right)^3$ is defined as in (4.2) and (4.3). Then by using the result of Lemma 5.1, we can easily prove the following theorem.

Theorem 5.1 *The functional $\mathcal{F}(\sigma, \alpha, \cdot)$ defined by (5.11) on $(H_0^1(\Omega))^2$ is strictly convex and coercive over the space $(Q_{1,0}(\Omega_h))^2$.*

PROOF: Because $\dot\sigma(U)$ is piecewise bilinear, and $\dot\epsilon(U)$ is piecewise linear over Ω, for any rectangular element $T_n \in \mathcal{T}_h$, we have that $(\dot\sigma(U), \dot\epsilon(U)) \in Q_2(T_n)$, where

$$Q_2(T_n) = \left\{ h(x, y) \;\middle|\; \begin{array}{r} h(x, y) = a_1 + a_2 x + a_3 y + a_4 x y + a_5 x^2 + a_6 x^2 y \\ + a_7 x y^2 + a_8 y^2 + a_9 x^2 y^2 \quad \forall (x, y) \in T_n \end{array} \right\} .$$

For the four points Gaussian quadrature, it is exact for any function in Q_2, *i.e.* for any $T_n \in \mathcal{T}_h$, we have

$$\int_{T_n} (\dot\sigma(U), \dot\epsilon(U))\,dx = \sum_{k=1}^{4} \omega_k (\dot\sigma(U(\hat x_k)), \dot\epsilon(U(\hat x_k))) \quad , \tag{5.12}$$

where $\hat x_k$ $(k = 1,2,3,4)$ are the Gaussian points of the rectangular T_n, and ω_k $(k = 1,2,3,4)$ are the weighting coefficients.

From Lemma 5.1, we know that the functional $G(U(\hat x_k)) = (\dot\sigma(U(\hat x_k)), \dot\epsilon(U(\hat x_k)))$ is strictly convex over $(Q_{1,0}(T_n))^2$. We also know that the weighting coefficients, ω_k $(k = 1,2,3,4)$, are all positive. Therefore for any $U, V \in (Q_{1,0}(\Omega_h))^2$, $\theta \in (0,1)$ and $W = \theta U + (1-\theta) V$, we have

$$\begin{aligned}
&\int_{T_n} (\dot\sigma(W), \dot\epsilon(W))\,dx = \sum_{k=1}^{4} \omega_k (\dot\sigma(W(\hat x_k)), \dot\epsilon(W(\hat x_k))) \\
&= \sum_{k=1}^{4} \omega_k (\dot\sigma(\theta U(\hat x_k) + (1-\theta) V(\hat x_k)), \dot\epsilon(\theta U(\hat x_k) + (1-\theta) V(\hat x_k))) \\
&< \sum_{k=1}^{4} \omega_k \theta (\dot\sigma(U(\hat x_k)), \dot\epsilon(U(\hat x_k))) + \sum_{k=1}^{4} \omega_k (1-\theta)(\dot\sigma(V(\hat x_k)), \dot\epsilon(V(\hat x_k))) \\
&= \theta \int_{T_n} (\dot\sigma(U), \dot\epsilon(U))\,dx + (1-\theta) \int_{T_n} (\dot\sigma(V), \dot\epsilon(V))\,dx \quad ,
\end{aligned}$$

for any $T_n \in \mathcal{T}_h$. Obviously $\int_\Omega \dot f \dot U\,dx$ is a linear functional of $\dot U$. So $\mathcal{F}(\sigma, \alpha, \cdot)$ is strictly convex over $(Q_{1,0}(\Omega_h))^2$.

To prove the coerciveness, take a sequence $\{\dot U_n\}_{n=1}^{\infty} \in (Q_{1,0}(\Omega_h))^2$, such that $\| \dot U_n \|_{H^1(\Omega)} \longrightarrow \infty$ as $n \longrightarrow \infty$. Then from Proposition 2.1, we know that there exists a constant γ'

$$(\dot\sigma(U(\hat x_k)), \dot\epsilon(U(\hat x_k))) \geq \gamma' (\dot\epsilon(U(\hat x_k)), \dot\epsilon(U(\hat x_k))) \qquad k = 1,2,3,4,$$

where γ' is independent of the value of $\sigma, \alpha, \dot\epsilon$ and $U(\hat x_k)$.

Hence,

$$\begin{aligned}
\mathcal{F}(\sigma,\alpha,\dot U_n) &= \frac{1}{2}\int_\Omega(\dot\sigma(U_n),\dot\epsilon(U_n))\,dx - \int_\Omega(\dot f,\dot U_n)\,dx \\
&= \frac{1}{2}\sum_{T_n\in\mathcal{T}_h}\sum_{k=1}^{4}\omega_k(\dot\sigma(U_n(\hat x_k)),\dot\epsilon(U_n(\hat x_k))) - \int_\Omega(\dot f,\dot U_n)\,dx \\
&\geq \frac{1}{2}\sum_{T_n\in\mathcal{T}_h}\sum_{k=1}^{4}\omega_k\tilde\gamma(\dot\epsilon(U_n(\hat x_k)),\dot\epsilon(U_n(\hat x_k))) - \int_\Omega(\dot f,\dot U_n)\,dx \\
&= \frac{\tilde\gamma}{2}\int_\Omega(\dot\epsilon(U_n),\dot\epsilon(U_n))\,dx - \int_\Omega(\dot f,\dot U_n)\,dx \\
&\geq \left(\frac{c\tilde\gamma}{2}\|\dot U_n\|_{H^1(\Omega)} - \|\dot f\|_{L^2(\Omega)}\right)\|\dot U_n\|_{H^1(\Omega)} \quad ,
\end{aligned}$$

by using the well known Korn's Inequality (see, e.g. [4] [13] [8]). Hence as $\|\dot U_n\|_{H^1(\Omega)} \longrightarrow \infty$, we have $\mathcal{F}(\sigma,\alpha,\dot U_n) \longrightarrow \infty$, and the coerciveness is proved. □

Now let us consider the following minimization problem

$$\min_{\dot U\in H_h}\mathcal{F}(\sigma,\alpha,\dot U) = \min_{\dot U\in H_h}\frac{1}{2}\int_\Omega(\dot\sigma(U),\dot\epsilon(U))\,dx - \int_\Omega \dot f\dot U\,dx \tag{5.13}$$

where $H_h = (Q_{1,0}(\Omega_h))^2$. Then from the previous theorem, we know that for any $(\sigma,\alpha) \in (\tilde Q_1(\Omega_h))^3\times(\tilde Q_1(\Omega_h))^m$, $\mathcal{F}(\sigma,\alpha,\dot U)$ is a strictly convex and coercive functional of $\dot U \in H_h$. So the minimization problem (5.13) must have a unique solution $\dot U \in H_h$. Therefore we can have the following corollary:

Corollary 5.1 *The functional $\mathcal{F}(\sigma,\alpha,\cdot)$ defined in (5.11) has a unique minimum over the space $(Q_{1,0}(\Omega_h))^2$.*

By showing that the functional $\mathcal{F}(\sigma,\alpha,\cdot)$ is Fréchet differentiable on the space $(Q_{1,0}(\Omega_h))^2$, we then have the following theorem:

Theorem 5.2 *The minimum $\dot U$ of (5.13) is the solution of the following problem:*

$$\begin{cases}
\int_\Omega(\dot\sigma_{11},\frac{\partial\phi_n}{\partial x_1}) + (\dot\sigma_{12},\frac{\partial\phi_n}{\partial x_2}) = \int_\Omega(\dot f_1,\phi_n) \\
\\
\int_\Omega(\dot\sigma_{21},\frac{\partial\phi_n}{\partial x_1}) + (\dot\sigma_{22},\frac{\partial\phi_n}{\partial x_2}) = \int_\Omega(\dot f_2,\phi_n)
\end{cases} \qquad n\in\mathcal{N} \tag{5.14}$$

with the constitutive equations:

$$\begin{cases}
\dot\sigma(\hat x_k) = D\dot\epsilon(\hat x_k) \\
\\
\dot\alpha(\hat x_k) = 0
\end{cases} \quad if \quad \begin{cases}
F(\sigma(\hat x_k),\alpha(\hat x_k)) < Z_0 \\
\quad or \\
\partial_\sigma F^T(\hat x_k)\dot\sigma(\hat x_k) \leq 0
\end{cases} \tag{5.15}$$

$$\begin{cases}
\dot\sigma(\hat x_k) = (D-D')(\hat x_k)\,\dot\epsilon(\hat x_k) \\
\\
\dot\alpha(\hat x_k) = -\dfrac{\partial_\sigma F^T(\hat x_k)\dot\sigma(\hat x_k)}{(\partial_\alpha F^T\partial_\alpha F)(\hat x_k)}\partial_\alpha F(\hat x_k)
\end{cases} \quad if \quad \begin{cases}
F(\sigma(\hat x_k),\alpha(\hat x_k)) = Z_0 \\
\quad and \\
\partial_\sigma F^T(\hat x_k)\dot\sigma(\hat x_k) > 0
\end{cases} \tag{5.16}$$

here $\hat x_k \in \hat G$ is any Gaussian point in the domain Ω , and $D'(\hat x_k)$ is defined as in (4.4).

PROOF: Let $U, \phi \in (Q_{1,0}(\Omega_h))^2$, $\nu > 0$ and $W = U + \nu\phi$. Denote

$$H_\nu = \int_{T_n} \left[\frac{(\dot\sigma(W_\nu), \dot\epsilon(W_\nu)) - (\dot\sigma(U), \dot\epsilon(U))}{2\nu} - (\dot\sigma(U), \dot\epsilon(\phi)) \right] dx$$

$$= \sum_{k=1}^{4} \omega_k \left[\frac{(\dot\sigma(W_\nu(\hat x_k)), \dot\epsilon(W_\nu(\hat x_k))) - \dot\sigma(U(\hat x_k)), \dot\epsilon(U(\hat x_k)))}{2\nu} - (\dot\sigma(U(\hat x_k)), \dot\epsilon(\phi(\hat x_k))) \right] ,$$

and

$$h_\nu(\hat x) = \frac{(\dot\sigma(W_\nu(\hat x)), \dot\epsilon(W_\nu(\hat x))) - (\dot\sigma(U(\hat x)), \dot\epsilon(U(\hat x)))}{2\nu} - (\dot\sigma(W_\nu(\hat x)), \dot\epsilon(\phi(\hat x))) .$$

We will discuss the following five different cases:

$$\hat x_k \in \mathcal{E}_0 = \left\{ x \in \widehat{G} \mid F(\sigma(x), \alpha(x)) < Z_0 \right\}$$

$$\hat x_k \in \mathcal{P}_1^\nu = \left\{ x \in \widehat{G} \mid F(\sigma(x), \alpha(x)) = Z_0 , \begin{array}{l} \partial_\sigma F^T D \dot\epsilon(U)(x) > 0 \\ \partial_\sigma F^T D \dot\epsilon(W_\nu)(x) > 0 \end{array} \right\}$$

$$\hat x_k \in \mathcal{P}_2^\nu = \left\{ x \in \widehat{G} \mid F(\sigma(x), \alpha(x)) = Z_0 , \begin{array}{l} \partial_\sigma F^T D \dot\epsilon(U)(x) \le 0 \\ \partial_\sigma F^T D \dot\epsilon(W_\nu)(x) \le 0 \end{array} \right\}$$

$$\hat x_k \in \mathcal{P}_3^\nu = \left\{ x \in \widehat{G} \mid F(\sigma(x), \alpha(x)) = Z_0 , \begin{array}{l} \partial_\sigma F^T D \dot\epsilon(U)(x) > 0 \\ \partial_\sigma F^T D \dot\epsilon(W_\nu)(x) \le 0 \end{array} \right\}$$

$$\hat x_k \in \mathcal{P}_4^\nu = \left\{ x \in \widehat{G} \mid F(\sigma(x), \alpha(x)) = Z_0 , \begin{array}{l} \partial_\sigma F^T D \dot\epsilon(U)(x) \le 0 \\ \partial_\sigma F^T D \dot\epsilon(W_\nu)(x) > 0 \end{array} \right\}$$

1). On the set $\mathcal{E}_0 \bigcup \mathcal{P}_2^\nu$, we have

$$h_\nu(\hat x) = \frac{(D\dot\epsilon(U + \nu\phi), \dot\epsilon(U + \nu\phi)) - (D\dot\epsilon(U), \dot\epsilon(U))}{2\nu} - (D\dot\epsilon(U + \nu\phi), \dot\epsilon(\phi)) ,$$

or

$$h_\nu(\hat x) = -\frac{\nu}{2}(D\dot\epsilon(\phi), \dot\epsilon(\phi)) . \tag{5.17}$$

2). Similarly on the set $\mathcal{P}_1^\nu$, we have

$$h_\nu(\hat x) = -\frac{\nu}{2}\left(\left(D - \frac{D\partial_\sigma F \partial_\sigma F^T D}{\partial_\alpha F^T \partial_\sigma F + \partial_\sigma F^T D \partial_\sigma F} \right) \dot\epsilon(\phi), \dot\epsilon(\phi) \right) . \tag{5.18}$$

3). On the set $\mathcal{P}_3^\nu$, we can use the decomposition (2.15), (2.16), (2.17) and get

$$h_\nu(\hat x) = \frac{1}{2\nu}\left(\langle \dot\epsilon(W_\nu), \dot\epsilon(W_\nu) \rangle - \langle \dot\epsilon(U) - \mu \partial_\sigma F^T D \dot\epsilon(U) \partial_\sigma F , \dot\epsilon(U) \rangle \right)$$

$$-\langle \dot\epsilon(U) - \mu\partial_\sigma F^T D\dot\epsilon(U)\partial_\sigma F, \dot\epsilon(\phi)\rangle$$

$$= \frac{1}{2\nu}(\langle\dot\epsilon(W_\nu),\dot\epsilon(W_\nu)\rangle - \langle\dot\epsilon(U),\dot\epsilon(U)\rangle) - \langle\dot\epsilon(U),\dot\epsilon(\phi)\rangle$$

$$+\frac{\gamma}{2\nu}\langle\mu\partial_\sigma F,\dot\epsilon(U)\rangle + \gamma\langle\mu\partial_\sigma F,\dot\epsilon(\phi)\rangle \quad ,$$

here $\gamma = \partial_\sigma F^T D\dot\epsilon(U)$. So we get

$$h_\nu(\hat x) = \frac{\nu}{2}\langle\dot\epsilon(\phi),\dot\epsilon(\phi)\rangle + \frac{\gamma^2}{2\nu}\langle\mu\partial_\sigma F, \frac{\partial_\sigma F}{\langle\partial_\sigma F,\partial_\sigma F\rangle}\rangle$$
$$+\gamma\langle\mu\partial_\sigma F,\dot\epsilon(\phi)\rangle \quad . \tag{5.19}$$

From the definition, we know that on $\mathcal{P}_3^\nu$ we have

$$\begin{cases} \partial_\sigma F^T D\dot\epsilon(U) > 0 \\ \partial_\sigma F^T D\dot\epsilon(U) + \nu\partial_\sigma F^T D\dot\epsilon(\phi) \le 0 \end{cases}$$

Hence $|\gamma| = |\partial_\sigma F^T D\dot\epsilon(U)| \le \nu|\partial_\sigma F^T D\dot\epsilon(\phi)|$, and (5.19) yields

$$h_\nu(\hat x) \le \frac{\nu}{2}\langle\dot\epsilon(\phi),\dot\epsilon(\phi)\rangle + \frac{\nu}{2}|\partial_\sigma F^T D\dot\epsilon(\phi)|^2\langle\mu\partial_\sigma F, \frac{\partial_\sigma F}{\langle\partial_\sigma F,\partial_\sigma F\rangle}\rangle$$
$$+\nu|\partial_\sigma F^T D\dot\epsilon(\phi)|\cdot|\langle\mu\partial_\sigma F,\dot\epsilon(\phi)\rangle| \tag{5.20}$$
$$= \frac{\nu}{2}\langle\dot\epsilon(\phi),\dot\epsilon(\phi)\rangle + \frac{3\nu}{2}\mu|\partial_\sigma F^T D\dot\epsilon(\phi)|^2 \quad .$$

3). On the set $\mathcal{P}_4^\nu$, we have

$$h_\nu(\hat x) = \frac{\langle\dot\epsilon(W_\nu) - \mu\partial_\sigma F^T D\dot\epsilon(W_\nu)\partial_\sigma F,\dot\epsilon(W_\nu)\rangle - \langle\dot\epsilon(U),\dot\epsilon(U)\rangle}{2\nu}$$

$$-\langle\dot\epsilon(U),\dot\epsilon(\phi)\rangle$$

$$= \frac{\nu}{2}\langle\dot\epsilon(\phi),\dot\epsilon(\phi)\rangle - \frac{\mu}{2\nu}\partial_\sigma F^T D\dot\epsilon(W_\nu)\langle\partial_\sigma F,\dot\epsilon(W_\nu)\rangle$$

$$= \frac{\nu}{2}\langle\dot\epsilon(\phi),\dot\epsilon(\phi)\rangle - \frac{\mu}{2\nu}\partial_\sigma F^T D\dot\epsilon(W_\nu)\langle\partial_\sigma F,\dot\epsilon(U)+\nu\dot\epsilon(\phi)\rangle \quad .$$

From the definition of set $\mathcal{P}_4^\nu$, we have

$$\begin{cases} \partial_\sigma F^T D\dot\epsilon(U) + \nu\partial_\sigma F^T D\dot\epsilon(\phi) > 0 \\ \partial_\sigma F^T D\dot\epsilon(U) \le 0 \end{cases}$$

which yields $|\partial_\sigma F^T D\dot\epsilon(U)| \le \nu|\partial_\sigma F^T D\dot\epsilon(\phi)|$. So finally we get

$$|h_\nu(\hat x)| \le \frac{\nu}{2}\langle\dot\epsilon(\phi),\dot\epsilon(\phi)\rangle + \frac{\mu}{2\nu}\langle\partial_\sigma F,\dot\epsilon(U)+\nu\dot\epsilon(\phi)\rangle^2$$
$$= \frac{\nu}{2}\langle\dot\epsilon(\phi),\dot\epsilon(\phi)\rangle + \frac{\mu}{2\nu}\Big(\langle\partial_\sigma F,\dot\epsilon(U)\rangle^2 +$$
$$2\nu\langle\partial_\sigma F,\dot\epsilon(U)\rangle\langle\partial_\sigma F,\dot\epsilon(\phi)\rangle + \nu^2\langle\partial_\sigma F,\dot\epsilon(\phi)\rangle^2\Big) \tag{5.21}$$
$$\le \frac{\nu}{2}\langle\dot\epsilon(\phi),\dot\epsilon(\phi)\rangle + 2\mu\nu\langle\partial_\sigma F,\dot\epsilon(\phi)\rangle^2 \quad .$$

Therefore using (5.17),(5.18),(5.20),(5.21) and the uniform boundedness of the gradient $\partial_\sigma F$, we can get that

$$|h_\nu(\hat{x}_k)| \le \nu C \langle \dot{\epsilon}(\phi(\hat{x}_k)), \dot{\epsilon}(\phi(\hat{x}_k)) \rangle \qquad \forall \phi \in H_h \text{ and } k = 1,2,3,4$$

for some $C > 0$.

Hence finally we get

$$\left| \frac{\mathcal{F}(\sigma,\alpha,W_\nu) - \mathcal{F}(\sigma,\alpha,U)}{\nu} - \left[\int_\Omega \dot{\sigma}(U)\dot{\epsilon}(\phi)\,dx - \int_\Omega \dot{f}\phi\,dx \right] \right|$$
$$\le \left| \sum_{k=1}^{4} \omega_k h_\nu(\hat{x}_k) \right| \le \sum_{k=1}^{4} \nu C \omega_k \langle \dot{\epsilon}(\phi(\hat{x}_k)), \dot{\epsilon}(\phi(\hat{x}_k)) \rangle$$
$$= \nu C \int_\Omega (D\dot{\epsilon}(\phi(x)), \dot{\epsilon}(\phi(x)))\,dx,$$

which means $\mathcal{F}(\sigma,\alpha,\cdot)$ is differentiable on $(Q_{1,0}(\Omega_h))^2$. Differentiating $\mathcal{F}$ with respect to $\dot{U}$, we get that the minimum $\dot{U}$ of (5.13) satisfies the system (5.14),(5.15) and (5.16). This completes the proof of the theorem. □

By using the results of Corollary 5.1 and Theorem 5.2, we can now prove the following theorem about existence and uniqueness of the solution of the semi-discrete problem.

Theorem 5.3 *If the body traction function* $\mathbf{f}(x,t)$ *is piecewise analytic in* t, *and for every fixed* $t \in [0,T]$, $\mathbf{f}(x,t) \in L^2(\Omega)$, *then there is a unique solution for the ODE system (4.1), (4.2), (4.3) and (4.4) satisfying the initial condition (4.5), and moreover the solution is piecewise analytic in time* t.

PROOF: We will prove this theorem in 4 steps. For the sake simplicity, we will assume that $\mathbf{f}(x,t)$ is analytic in t and $F(\sigma,\alpha)$ is analytic in σ and α. But the same result still holds when $\mathbf{f}$ and F are piecewise analytic.

Step 1. Here we will assume that at the beginning the material is in its virgin state. So at all the Gaussian points the material is elastic until a time $t = t_0$ when $F(\sigma,\alpha)(\hat{x}_k,t_0) = F(\sigma,0)(\hat{x}_k,t_0) = Z_0$ at some Gaussian points of some elements. Denote by $\widehat{G}_1$ the set of those Gaussian points and $\widehat{G}$ the set of all the Gaussian points of Ω_h. Then for $t \in [0,t_0)$, all the points are elastic, and we just have a classical linear elasticity problem. So the problem will have a unique analytic solution over the time interval $[0,t_0)$.

Step 2. As we will see in the following proof, the points of the set $\widehat{G}\backslash\widehat{G}_1$ will remain elastic after t_0. But those of $\widehat{G}_1$ may or may not yield after t_0. To determine this we have to check the sign of $\partial_\sigma F^T(\sigma,\alpha)\dot{\sigma}(\hat{x}_k,t_0+0)$. Consider an auxiliary equation system such that $(\sigma^1,U^1) \in (\tilde{Q}_1(\Omega_h))^3 \times (Q_{1,0}(\Omega_h))^2$ would solve at t_0+0.

$$\begin{cases} \int_\Omega (\sigma^1_{11}, \frac{\partial \phi_n}{\partial x_1}) + (\sigma^1_{12}, \frac{\partial \phi_n}{\partial x_2}) = \int_\Omega (\dot{f}_1(t_0+0), \phi_n) \\ \int_\Omega (\sigma^1_{21}, \frac{\partial \phi_n}{\partial x_1}) + (\sigma^1_{22}, \frac{\partial \phi_n}{\partial x_2}) = \int_\Omega (\dot{f}_2(t_0+0), \phi_n) \end{cases} \quad n \in \mathcal{N} \tag{5.22}$$

with the constitutive equations:

$$\sigma^1(\hat{x}_k) = D\epsilon^1(\hat{x}_k) \qquad \forall \hat{x}_k \in \hat{G}\backslash\hat{G}_1 \tag{5.23}$$

$$\begin{cases} \sigma^1(\hat{x}_k) = D\epsilon^1(\hat{x}_k) & \forall \hat{x}_k \in \hat{D}^1_- \\ \sigma^1(\hat{x}_k) = (D - D')\epsilon^1(\hat{x}_k) & \forall \hat{x}_k \in \hat{D}^1_+ \end{cases} \tag{5.24}$$

where

$$\hat{D}^1_- = \left\{ \hat{x} \in \hat{G}_1 \;\middle|\; \partial_\sigma F^T(t_0) D\epsilon(U^1)(\hat{x}) \le 0 \right\}$$

$$\hat{D}^1_+ = \left\{ \hat{x} \in \hat{G}_1 \;\middle|\; \partial_\sigma F^T(t_0) D\epsilon(U^1)(\hat{x}) > 0 \right\}$$

and

$$D' = D'(x, t_0) = \frac{D\partial_\sigma F \partial_\sigma F^T D}{\partial_\alpha F^T \partial_\alpha F + \partial_\sigma F^T D \partial_\sigma F}(x, t_0) \quad .$$

From Corollary 5.1 and Theorem 5.2, we know that there exists a unique solution (σ^1, U^1) of the equation system (5.22)$\sim$(5.24).

Step 3. Now we can define the sets

$$\hat{G}^p_1 = \left\{ \hat{x} \in \hat{G}_1 \;\middle|\; \partial_\sigma F^T(\hat{x}, t_0)\sigma^1(\hat{x}) > 0 \right\}$$

$$\hat{G}^e_1 = \left\{ \hat{x} \in \hat{G}_1 \;\middle|\; \partial_\sigma F^T(\hat{x}, t_0)\sigma^1(\hat{x}) \le 0 \right\}$$

and consider the following initial value problem starting at $t = t_0$:

$$\begin{cases} \int_\Omega (\dot{\sigma}_{11}, \frac{\partial \phi_n}{\partial x_1}) + (\dot{\sigma}_{12}, \frac{\partial \phi_n}{\partial x_2}) = \int_\Omega (\dot{f}_1(t), \phi_n) \\ \int_\Omega (\dot{\sigma}_{21}, \frac{\partial \phi_n}{\partial x_1}) + (\dot{\sigma}_{22}, \frac{\partial \phi_n}{\partial x_2}) = \int_\Omega (\dot{f}_2(t), \phi_n) \end{cases} \quad n \in \mathcal{N} \tag{5.25}$$

$$\begin{cases} \dot{\sigma}(\hat{x}_k) = D\dot{\epsilon}(\hat{x}_k) \\ \dot{\alpha}(\hat{x}_k) = 0 \end{cases} \qquad \forall \hat{x}_k \in \hat{G}\backslash\hat{G}^p_1 \tag{5.26}$$

$$\begin{cases} \dot{\sigma}(\hat{x}_k) = (D - D')\dot{\epsilon}(\hat{x}_k) \\ \dot{\alpha}(\hat{x}_k) = \dfrac{-\partial_\sigma F^T(\hat{x}_k)\dot{\sigma}(\hat{x}_k)}{\partial_\alpha F^T(\hat{x}_k)\partial_\alpha F(\hat{x}_k)}\partial_\alpha F(\hat{x}_k) \end{cases} \qquad \forall \hat{x}_k \in \hat{G}^p_1 \tag{5.27}$$

with

$$\dot{\epsilon}(\hat{x}_k) = \epsilon(\dot{U}(\hat{x}_k))$$

$$D'(\hat{x}_k) = \frac{D\partial_\sigma F(\hat{x}_k)\partial_\sigma F^T(\hat{x}_k) D}{\partial_\alpha F^T(\hat{x}_k)\partial_\alpha F(\hat{x}_k) + \partial_\sigma F^T(\hat{x}_k) D \partial_\sigma F(\hat{x}_k)}$$

and the initial conditions:

$$(U, \sigma, \alpha)(\hat{x}_k, t_0) = (U, \sigma, \alpha)(\hat{x}_k, t_0 - 0) \quad . \tag{5.28}$$

Now we can show that the initial value problem (5.25)~(5.28) has a unique analytic solution (U, σ, α) on the interval $[t_0, t_0 + \delta)$ with some $\delta > 0$, and moreover

$$(\dot{\mathrm{U}}, \dot{\sigma})|_{t_0+0} = (\mathrm{U}^1, \sigma^1)$$

where (U^1, σ^1) is the solution of the problem (5.22) ~(5.24).

Indeed, substituting (5.26) and (5.27) into (5.25), because of the assumption that the gauge function $F(\sigma, \alpha)$ is an analytic function of σ and α, we will get an equation in $\dot{U}$ with analytic coefficients. On the other hand from Proposition 2.1, we know that D and $D - D'$ are positive definite matrices. Therefore there exists a unique solution $\dot{U}$ which is analytic on a interval $[t_0, t_0 + \delta)$ for some $\delta > 0$. For the stress and the hardening parameter functions (σ, α), again because of the analytic assumption of the gauge function $F(\sigma, \alpha)$, we can write (5.26) and (5.27) as a system of ODE's in the form

$$\frac{d}{dt}\begin{bmatrix} \sigma \\ \alpha \end{bmatrix} = A(\sigma, \alpha, t) \tag{5.29}$$

where $A(\sigma, \alpha, t)$ is an analytic function of σ, α and t. Therefore (5.29) would have a unique analytic solution over the interval $[t_0, t_0 + \delta)$ satisfying the initial conditions (5.28).

The solution of problem (5.22)~(5.24), (U^1, σ^1), satisfies the following problem at $t_0 + 0$

$$\begin{cases} \int_\Omega (\sigma^1_{11}, \frac{\partial \phi_n}{\partial x_1}) + (\sigma^1_{12}, \frac{\partial \phi_n}{\partial x_2}) = \int_\Omega (\dot{f}_1(t_0 + 0), \phi_n) \\ \int_\Omega (\sigma^1_{21}, \frac{\partial \phi_n}{\partial x_1}) + (\sigma^1_{22}, \frac{\partial \phi_n}{\partial x_2}) = \int_\Omega (\dot{f}_2(t_0 + 0), \phi_n) \end{cases} \quad n \in \mathcal{N} \tag{5.30}$$

where

$$\begin{cases} \sigma^1(\hat{x}_k) = D\epsilon^1(\hat{x}_k) & \forall \hat{x} \in \widehat{G}\backslash\widehat{G}^p_1 \\ \sigma^1(\hat{x}_k) = (D - D')\epsilon^1(\hat{x}_k) & \forall \hat{x} \in \widehat{G}^p_1 \end{cases} \tag{5.31}$$

with $U^1 \in (Q_{1,0}(\Omega_h))^2$. By virtue of Theorem 5.2, this problem has a unique solution. On the other hand, $(\dot{U}, \dot{\sigma})$, the derivative of the solution of problem (5.25)~(5.28) with respect to time t also satisfies (5.30) ~ (5.31) at $t_0 + 0$, therefore we have $(\dot{U}, \dot{\sigma})|_{t_0+0} = (U^1, \sigma^1)$.

Step 4. In this step we will show that there exists a constant $\tilde{\delta} > 0$, such that at any Gaussian point $\hat{x}_k$, if it is elastic (or plastic) at the time $t = t_0 + 0$, then it will still be elastic (or plastic) over the time interval $[t_0, t_0 + \tilde{\delta})$.

a). For any Gaussian point $\hat{x}_k \in \hat{G}\backslash\hat{G}_1 = \left\{ \hat{x} \in \hat{G} \;\mid\; F(\sigma^1,\alpha^1)(\hat{x}) < Z_0 \right\}$, because the solution (U,σ,α) is analytic over $[t_0, t_0+\delta)$, so we can always find a constant $0 < \tilde{\delta}_1 \le \delta$ such that

$$F(\sigma,\alpha)(\hat{x}_k,t) < Z_0 \qquad \forall\, \hat{x}_k \in \hat{G}\backslash\hat{G}_1 \quad \text{over the interval } [t_0, t_0+\tilde{\delta}_1)$$

which implies that the Gaussian points in $\hat{G}\backslash\hat{G}_1$ will remain in elastic state over the time interval $[t_0, t_0+\tilde{\delta}_1)$.

b). For any Gaussian point $\hat{x}_k \in \hat{G}$ such that $F(\sigma^1,\alpha^1)(\hat{x}_k) = Z_0$ and $\partial_\sigma F^T(\hat{x}_k,t_0)\sigma^1(\hat{x}_k) < 0$, we have

$$F^2(\sigma,\alpha)(\hat{x}_k,t_0+\nu) - F^2(\sigma,\alpha)(\hat{x}_k,t_0) = \frac{1}{2}\int_{t_0}^{t_0+\nu} F(\sigma,\alpha)(\partial_\sigma F^T\dot{\sigma} + \partial_\alpha F^T\dot{\alpha})\,dt \quad .$$

So if $\nu \le \delta$ is small enough, we have $\partial_\sigma F^T\dot{\sigma}(\hat{x}_k,t) < 0$ and $\dot{\alpha}(\hat{x}_k,t) = 0$ for any $t \in [t_0, t_0+\nu)$. (from the analytic assumption of the gauge function $F(\sigma,\alpha)$ and the analyticity of the solution functions (U,σ,α) and the fact that $\dot{\sigma}(\hat{x}_k,t_0+0) = \sigma^1(\hat{x}_k)$) Therefore

$$F^2(\sigma,\alpha)(\hat{x}_k,t_0+\nu) - F^2(\sigma,\alpha)(\hat{x}_k,t_0) = \frac{1}{2}\int_{t_0}^{t_0+\nu} F(\sigma,\alpha)\partial_\sigma F^T\dot{\sigma}\,dt < 0 \quad ,$$

i.e.

$$F^2(\sigma,\alpha)(\hat{x}_k,t_0+\nu) < F^2(\sigma,\alpha)(\hat{x}_k,t_0) = Z_0^2 \quad .$$

Hence for $t \in [t_0, t_0+\tilde{\delta}_2)$ we have $F(\sigma,\alpha)(\hat{x}_k,t) < Z_0$, which means that the elastic Gaussian points of $\hat{G}_1^e$ remain elastic over the time interval $[t_0, t_0+\tilde{\delta}_2)$.

c). For any $\hat{x}_k \in \hat{G}_1^p$, we also have:

$$F^2(\sigma,\alpha)(\hat{x}_k,t_0+\nu) - F^2(\sigma,\alpha)(\hat{x}_k,t_0) = \frac{1}{2}\int_{t_0}^{t_0+\nu} F(\sigma,\alpha)(\partial_\sigma F^T\dot{\sigma} + \partial_\alpha F^T\dot{\alpha})\,dt \quad .$$

Since on $\hat{G}_1^p$ we have $\partial_\sigma F^T\dot{\sigma}(\hat{x}_k,t_0+0) > 0$, once more by the analyticity of the gauge function $F(\sigma,\alpha)$ and the continuity of the solution σ, α, $\dot{\sigma}$ and $\dot{\alpha}$, we know that on $[t_0, t_0+\nu)$, for ν small enough, we have $\partial_\sigma F^T\dot{\sigma} > 0$, and moreover $\dot{\alpha}$ cannot vanish, so $F(\sigma,\alpha)(\partial_\sigma F^T\dot{\sigma} + \partial_\alpha F^T\dot{\alpha}) = 0$. Hence we have $F(\sigma,\alpha)(\hat{x}_k,t) = F(\sigma,\alpha)(\hat{x}_k,t_0) = Z_0$, $\forall\, t \in [t_0, t_0+\tilde{\delta}_3)$, which means that the plastic Gaussian points of G_1^p remain plastic.

So if we denote by $\hat{G}_2$ the set:

$$\hat{G}_2 = \left\{ \hat{x} \in \hat{G}_1 \;\mid\; \partial_\sigma F^T(\hat{x})\sigma^1(\hat{x}) = 0 \right\} \quad ,$$

and if G_2 is empty, then from the above discussion we can see that the elastic Gaussian point set and the plastic Gaussian point set determined by the solution (U^1,σ^1,α^1) of problem (5.22)$\sim$(5.24) will remain unchanged for a certain period of time $[t_0, t_0+\tilde{\delta})$. Thus whenever the elastic and the plastic Gaussian point sets were determined at the time level $t = t_0+0$, we can always solve the initial value problem (5.14)$\sim$(5.16) starting at $t = t_0$ and forward to the time level $t_0+\tilde{\delta}$ without violating the admissible conditions $F(\sigma,\alpha)(\hat{x}_k,t) \le Z_0$.

Remark 5.1 If $\widehat{G}_2$ is not empty, we can show that the choice for the Gaussian points of $\widehat{G}_2$ will not change the state of the points of $\widehat{G}\backslash\widehat{G}_2$. By considering the sign of higher order derivatives of the function $\partial_\sigma F^T(\hat{x}_k, t_0) D\dot{\epsilon}(\hat{x}_k, t_0+0)$, we can also show that there exists a positive number ν, such that for $t \in [t_0, t_0+\nu)$, the elastic Gaussian points remain elastic and the plastic Gaussian points remain plastic. The details of the proof can be obtained by slightly modifying the proof in Bonnetier's thesis [3].

In conclusion, we can always determine the state of each Gaussian point of the set $\widehat{G}$ for a certain period of time after t_0 by only knowing the history of the solution before t_0 and the information about the traction function $\dot{f}(x, t_0+0)$. Therefore the problem (5.25)$\sim$(5.28) always has a unique analytic solution over some time interval $[t_0, t_0+\delta)$ with the sets $\widehat{G}_1^p$ and $\widehat{G}\backslash\widehat{G}_1^p$ being modified a little bit if the set $\widehat{G}_2$ is not empty. Thus we can always solve an initial value problem starting at $t = t_i$ and forward to another time level $t = t_{i+1}$ when we have some Gaussian points switch their elastic or plastic state. Then we will use the same method to determine the elastic or plastic state for each Gaussian point for the time $t = t_{i+1}+0$, and solve another initial value problem starting at $t = t_{i+1}$. The similar results still hold for the subsequent yieldings. Hence finally we will get a unique solution, piecewise analytic in time, of the system (4.1)$\sim$(4.4) over the time interval $[0, T]$ satisfying the initial value condition (4.5). This completes our proof of the theorem. $\square$

6 Energy Estimates

To prove the convergence of the solution for the semi-discrete problem, we need the uniform boundedness of the solution. By using the admissible assumptions, we can have the following energy estimate.

Theorem 6.1 *There is a positive constant C independent of the finite element mesh, such that*

$$\| \dot{U} \|_{H_0^1(\Omega)}, \ \| \dot{\epsilon} \|_{L^2(\Omega)}, \ \| \dot{\sigma} \|_{L^2(\Omega)}, \ \| \dot{\alpha} \|_{L^2(\Omega)} \ \leq \ C \| \dot{f} \|_{L^2(\Omega)}$$

a.e. $t \in (0, T)$.

Proof: Let (U, σ, α) be a $C_+^{0,\infty}$ solution of (4.1)$\sim$(4.4) with the initial condition (4.5) in the interval $[0, T)$. Then at any Gaussian point $\hat{x}_k$, from Proposition 2.1, we know that there exists a constant c independent of the element mesh such that

$$c\,(\dot{\epsilon}(\hat{x}_k), \dot{\epsilon}(\hat{x}_k)) \leq (\dot{\sigma}(\hat{x}_k), \dot{\epsilon}(\hat{x}_k)) \qquad \forall\, \hat{x}_k \in G \ .$$

So

$$c\int_T (\dot{\epsilon}, \dot{\epsilon})\, dx = c\sum_{k=1}^{4} \omega_k(\dot{\epsilon}(\hat{x}_k), \dot{\epsilon}(\hat{x}_k)) \leq \sum_{k=1}^{4} \omega_k(\dot{\sigma}(\hat{x}_k), \dot{\epsilon}(\hat{x}_k)) = \int_T (\dot{\sigma}, \dot{\epsilon})\, dx \ .$$

By virtue of (4.1), we get

$$c\int_\Omega (\dot{\epsilon}, \dot{\epsilon})\, dx \leq \int_\Omega (\dot{\sigma}, \dot{\epsilon})\, dx = \int_\Omega (\dot{f}, \dot{U})\, dx \ . \tag{6.1}$$

Using Korn's inequality, we have

$$\| \nabla \dot{U} \|^2_{L^2(\Omega)} \leq c \| \dot{f} \|_{L^2(\Omega)} \| \dot{U} \|_{L^2(\Omega)} \quad .$$

So we finally get

$$\| \dot{U} \|_{H^1_0(\Omega)} \leq c_1 \| \dot{f} \|_{L^2(\Omega)} \quad . \tag{6.2}$$

From (6.1) and (6.2), we have

$$\| \dot{\epsilon} \|^2_{L^2(\Omega)} \leq C \| \dot{f} \|_{L^2(\Omega)} \| \dot{U} \|_{L^2(\Omega)} \leq C_1 \| \dot{f} \|^2_{L^2(\Omega)} \quad .$$

For the boundedness of the stress, we have

$$\dot{\epsilon}(\hat{x}_k) = D^{-1} \dot{\sigma}(\hat{x}_k) \qquad \text{if } \hat{x}_k \text{ is elastic} \quad ,$$

or

$$\dot{\epsilon}(\hat{x}_k) = (D - D')^{-1} \dot{\sigma}(\hat{x}_k) \qquad \text{if } \hat{x}_k \text{ is plastic} \quad ,$$

where both D^{-1} and $(D-D')^{-1}$ are positive definite, and their smallest eigenvalue is bounded below by a positive constant c_2 independent of the value of t, σ, α and the element mesh. So we can use the same argument as the one we used before, and get

$$c_2 \int_\Omega (\dot{\sigma}, \dot{\sigma})\, dx \leq \int_\Omega (\dot{f}, \dot{U})\, dx \quad .$$

So

$$\| \dot{\sigma} \|^2_{L^2(\Omega)} \leq \frac{1}{c_2} \| \dot{f} \|_{L^2(\Omega)} \| \dot{U} \|_{L^2(\Omega)} \leq C_2 \| \dot{f} \|^2_{L^2(\Omega)} \quad , \tag{6.3}$$

by using the inequality (6.2).

For the boundedness of the hardening parameter, $\| \dot{\alpha} \|_{L^2(\Omega)}$, we can use the fact that

$$\dot{\alpha}(\hat{x}_k) = 0 \qquad \text{if } \hat{x}_k \text{ is elastic} \quad ,$$

or

$$\dot{\alpha}(\hat{x}_k) = - \frac{\partial_\sigma F^T(\hat{x}_k)\, \dot{\sigma}(\hat{x}_k)}{\partial_\alpha F^T(\hat{x}_k)\, \partial_\alpha F(\hat{x}_k)} \partial_\alpha F(\hat{x}_k) \qquad \text{if } \hat{x}_k \text{ is plastic} \quad .$$

So

$$[\dot{\alpha}(\hat{x}_k)]^2 \leq \left[\frac{\partial_\sigma F^T(\hat{x}_k)\, \dot{\sigma}(\hat{x}_k)}{\partial_\alpha F^T(\hat{x}_k)\, \partial_\alpha F(\hat{x}_k)} \partial_\alpha F(\hat{x}_k) \right]^2 \qquad \forall\, \hat{x}_k \in G \quad .$$

Using the admissible condition $\gamma \leq |\partial_\sigma F|, |\partial_\alpha F| \leq \Gamma$, we then have

$$[\dot{\alpha}(\hat{x}_k)]^2 \leq C\, [\dot{\sigma}(\hat{x}_k)]^2 \quad ,$$

that yields

$$\int_T (\dot{\alpha}, \dot{\alpha}) dx = \sum_{k=1}^{4} \omega_k (\dot{\alpha}(\hat{x}_k), \dot{\alpha}(\hat{x}_k)) \leq C \sum_{k=1}^{4} \omega_k (\dot{\sigma}(\hat{x}_k), \dot{\sigma}(\hat{x}_k)) = C \int_T (\dot{\sigma}, \dot{\sigma})\, dx \quad .$$

By virtue of (6.3), it follows that

$$\int_\Omega (\dot{\alpha}, \dot{\alpha})\, dx \leq C \int_\Omega (\dot{\sigma}, \dot{\sigma})\, dx \leq \hat{C} \int_\Omega (\dot{f}, \dot{f})\, dx \quad .$$

Hence $\| \dot{\alpha} \|_{L^2(\Omega)} \leq C \| \dot{f} \|_{L^2(\Omega)}$ and the proof is completed. □

7 An Equivalent Weak Formulation

The following weak formulation is the key point of our proof for the convergence of the semi-discrete problem solutions. Based on this formulation, we will be able to identify the equations satisfied by the limit of the semi-discrete problem solutions.

Denote the set

$$K_h = \left\{ (\tau_h, \nu_h) \;\middle|\; \begin{array}{l} \tau_h = \sum_{n \in \tilde{\mathcal{N}}} \tau_n(t)\,\tilde{\phi}_n(x)\,,\ \tau_n(t) \in C^1_+(I) \\ \nu_h = \sum_{n \in \tilde{\mathcal{N}}} \nu_n(t)\,\tilde{\phi}_n(x)\,,\ \nu_n(t) \in C^1_+(I) \end{array} \ \text{with} \ \begin{array}{c} F(\tau_h, \nu_h)(\hat{x}_k) \le Z_0 \\ \forall\, \hat{x}_k \in \hat{G} \end{array} \right\}$$

Consider then the functions of the following forms

$$\left\{ \begin{array}{lll} U_{h_i} = \sum_{n \in \mathcal{N}} U_i^n(t)\,\phi_n(x) & & i = 1,2 \\ \epsilon_{h_{11}} = \sum_{n \in \mathcal{N}} U_1^n(t)\,\dfrac{\partial \phi_n(x)}{\partial x_1} & & \\ \epsilon_{h_{22}} = \sum_{n \in \mathcal{N}} U_2^n(t)\,\dfrac{\partial \phi_n(x)}{\partial x_2} & & \\ \epsilon_{h_{12}} = \sum_{n \in \mathcal{N}} \left(U_2^n(t)\,\dfrac{\partial \phi_n(x)}{\partial x_1} + U_1^n(t)\,\dfrac{\partial \phi_n(x)}{\partial x_2} \right) & & \\ \sigma_{h_{ij}} = \sum_{n \in \tilde{\mathcal{N}}} \sigma_{ij}^n(t)\,\tilde{\phi}_n(x) & & i,j = 1,2 \\ \alpha_{h_i} = \sum_{n \in \tilde{\mathcal{N}}} \alpha_i^n(t)\,\tilde{\phi}_n(x) & & i = 1,2,\ldots,m \end{array} \right. \tag{7.1}$$

where $U_i^n(t), \sigma_{ij}^n(t), \alpha_i^n(t) \in C_+^{0,\infty}(I)$. we have the following theorem.

Theorem 7.1 *The semi - discrete initial value problem (4.1)~ (4.5) is equivalent to the following problem:*

Seek $(U_h, \sigma_h, \alpha_h)$ *of the form (7.1) such that for all* $t \in I$,

$$\left\{ \begin{array}{l} \int_\Omega \left(\dot{\sigma}_{h_{11}} \dfrac{\partial \phi_n}{\partial x_1} + \dot{\sigma}_{h_{12}} \dfrac{\partial \phi_n}{\partial x_2} \right) dx = \int_\Omega (\dot{f}_1, \phi_n)\, dx \\ \int_\Omega \left(\dot{\sigma}_{h_{21}} \dfrac{\partial \phi_n}{\partial x_1} + \dot{\sigma}_{h_{22}} \dfrac{\partial \phi_n}{\partial x_2} \right) dx = \int_\Omega (\dot{f}_2, \phi_n)\, dx \end{array} \right. \quad n \in \mathcal{N} \tag{7.2}$$

where

$$(\sigma_h, \alpha_h) \in K_h \quad , \quad U_h \in (H_0^1(\Omega))^2 \quad \forall\, t \in I \tag{7.3}$$

and

$$\int_\Omega (\dot{\epsilon}_h - C\,\dot{\sigma}_h, \tau_h - \sigma_h)\, dx + \int_\Omega (-\dot{\alpha}_h, \nu_h - \alpha_h)\, dx \le 0 \qquad \forall\, (\tau_h, \nu_h) \in K_h \tag{7.4}$$

with $C = D^{-1}$, *and the initial conditions*

$$\left\{ \begin{array}{l} U_h(0,x) = 0 \\ \sigma_h(0,x) = 0 \\ \alpha_h(0,x) = 0 \end{array} \right. \tag{7.5}$$

PROOF: i) First to show that the solution (U, σ, α) of the system (4.1)$\sim$ (4.5) satisfies (7.2)$\sim$(7.5). Indeed, we have for all $t \in I$,

$$F(\sigma(\hat{x}_k), \alpha(\hat{x}_k)) \leq Z_0 \qquad \forall \hat{x}_k \in \widehat{G} \quad ,$$

which means $(\sigma, \alpha) \in K_h$. Now suppose at the point $\hat{x}_k$ the material is elastic, then from (4.2), we have

$$\begin{cases} \dot{\epsilon}(\hat{x}_k) = C\dot{\sigma}(\hat{x}_k) \\ \dot{\alpha}(\hat{x}_k) = 0 \end{cases}$$

Obviously, in this case, we have

$$(\dot{\epsilon}(\hat{x}_k) - C\dot{\sigma}(\hat{x}_k), \tau(\hat{x}_k) - \sigma(\hat{x}_k)) + (-\dot{\alpha}(\hat{x}_k), \nu(\hat{x}_k) - \alpha(\hat{x}_k)) \leq 0 \quad . \tag{7.6}$$

If at the point $\hat{x}_k$ the material is plastic, then from (4.3), we know that there exists a constant $\lambda > 0$, such that

$$\begin{cases} \dot{\epsilon}^p(\hat{x}_k) = \lambda\, \partial_\sigma F(\sigma(\hat{x}_k), \alpha(\hat{x}_k)) \\ -\dot{\alpha}(\hat{x}_k) = \lambda\, \partial_\alpha F(\sigma(\hat{x}_k), \alpha(\hat{x}_k)) \end{cases} \tag{7.7}$$

where $\dot{\epsilon}^p(\hat{x}_k) = \dot{\epsilon}(\hat{x}_k) - C\dot{\sigma}(\hat{x}_k)$. On the other hand, since $F(\sigma, \alpha)$ is convex and analytic, we have (See, e.g. [18])

$$F(\tau, \nu) - F(\sigma, \alpha) \geq \partial_\sigma F(\sigma, \alpha)^T(\tau - \sigma) + \partial_\alpha F(\sigma, \alpha)^T(\nu - \alpha) \quad .$$

So if at the point $\hat{x}_k$ the material is plastic and $(\tau, \nu) \in K_h$, we must have

$$F(\tau(\hat{x}_k), \nu(\hat{x}_k)) - F(\sigma(\hat{x}_k), \alpha(\hat{x}_k)) = F(\tau(\hat{x}_k), \nu(\hat{x}_k)) - Z_0 \leq Z_0 - Z_0 = 0 \quad .$$

Hence

$$\partial_\sigma F(\sigma(\hat{x}_k), \alpha(\hat{x}_k))^T(\tau(\hat{x}_k) - \sigma(\hat{x}_k)) + \partial_\alpha F(\sigma(\hat{x}_k), \alpha(\hat{x}_k))^T(\nu(\hat{x}_k) - \alpha(\hat{x}_k)) \leq 0 \quad .$$

From (7.7), we can see that (7.6) still holds. Therefore on any $T_n \in \mathcal{T}_h$, we have

$$\int_{T_n} (\dot{\epsilon} - C\dot{\sigma}, \tau - \sigma)\, dx + \int_{T_n} (-\dot{\alpha}, \nu - \alpha)\, dx$$

$$= \sum_{k=1}^{4} \omega_k \left[(\dot{\epsilon}(\hat{x}_k) - C\dot{\sigma}(\hat{x}_k), \tau(\hat{x}_k) - \sigma(\hat{x}_k)) - (\dot{\alpha}(\hat{x}_k), \tau(\hat{x}_k) - \alpha(\hat{x}_k)) \right] \leq 0 \quad .$$

So

$$\int_\Omega (\dot{\epsilon} - C\dot{\sigma}, \tau - \sigma)\, dx + \int_\Omega (-\dot{\alpha}, \nu - \alpha)\, dx \leq 0 \qquad \forall (\tau, \nu) \in K_h \quad .$$

ii) Second to show that the solution of the system (7.2)$\sim$(7.5) is unique. Now suppose we have two solutions, say $(U_1, \sigma_1, \alpha_1)$ and $(U_2, \sigma_2, \alpha_2)$, then by the definition we have

$$(\sigma_1, \alpha_1) \in K_h \qquad \forall t \in I$$

$$(\sigma_2,\alpha_2) \in K_h \qquad \forall t \in I \quad .$$

By virtue of (7.4), it follows that

$$\int_\Omega ((\dot\epsilon_1 - \dot\epsilon_2) - C(\dot\sigma_1 - \dot\sigma_2), \sigma_2 - \sigma_1)\,dx + (-(\dot\alpha_1 - \dot\alpha_2), \alpha_2 - \alpha_1)\,dx \le 0 \quad ,$$

which implies

$$\int_\Omega (\dot\epsilon_1 - \dot\epsilon_2, \sigma_2 - \sigma_1)\,dx + \frac{1}{2}\frac{d}{dt}\left[\|\sqrt{C}\,(\sigma_2 - \sigma_1)\|^2\right] + \frac{1}{2}\frac{d}{dt}\left[\|\alpha_2 - \alpha_1\|^2\right] \le 0 \quad .$$

From (7.2), we can show that

$$\int_\Omega (\dot\epsilon_1 - \dot\epsilon_2, \sigma_2 - \sigma_1)\,dx = 0 \quad .$$

Therefore, we have

$$\frac{d}{dt}\left[\|\sqrt{C}\,(\sigma_2 - \sigma_1)\|^2 + \|\alpha_2 - \alpha_1\|^2\right] \le 0 \quad .$$

We already know that $(U_1, \sigma_1, \alpha_1)$ and $(U_2, \sigma_2, \alpha_2)$ satisfy the same initial conditions (7.5), so we get

$$\int_0^T \left[\|\sqrt{C}\,(\sigma_2 - \sigma_1)\|^2 + \|\alpha_2 - \alpha_1\|^2\right] dt \le 0 \quad ,$$

which means

$$\begin{cases} \sigma_1 = \sigma_2 \\ \alpha_1 = \alpha_2 \end{cases} \qquad a.e. \quad \Omega \times I \quad . \tag{7.8}$$

It follows that

$$\begin{cases} \dot\sigma_1 = \dot\sigma_2 \\ \dot\alpha_1 = \dot\alpha_2 \end{cases} \qquad a.e. \quad \Omega \times I \quad . \tag{7.9}$$

For the strain and displacement function, we know that at any Gaussian point $\hat{x}_k$, either

$$\dot\epsilon(\hat{x}_k) = D^{-1}\dot\sigma(\hat{x}_k) \qquad \text{if } \hat{x}_k \text{ is elastic}$$

or

$$\dot\epsilon(\hat{x}_k) = \left(D - \frac{D\partial_\sigma F(\sigma,\alpha)\partial_\sigma F(\sigma,\alpha)^T D}{\partial_\alpha F(\sigma,\alpha)^T \partial_\alpha F(\sigma,\alpha) + \partial_\sigma F(\sigma,\alpha)^T D \partial_\sigma F(\sigma,\alpha)}\right)^{-1}\Bigg|_{x=\hat{x}_k} \dot\sigma(\hat{x}_k)$$

if $\hat{x}_k$ is plastic. So by (7.8) and (7.9), we have

$$\dot\epsilon_1 = \dot\epsilon_2 \quad ,$$

and hence

$$U_1 = U_2 \quad .$$

iii) Finally to show that the solution of the system (7.2)$\sim$(7.5) also satisfies (4.1)$\sim$(4.5). Indeed, we have for all $t \in I$,

$$\int_\Omega (\dot\epsilon - C\dot\sigma, \tau - \sigma)\,dx + \int_\Omega (-\dot\alpha, \nu - \alpha)\,dx \le 0 \qquad \forall (\tau,\nu) \in K_h$$

So for any Gaussian point $\hat{x}_k$, take $\tau(\hat{x}) = \sigma(\hat{x})$, $\nu(\hat{x}) = \alpha(\hat{x})$ for all $\hat{x} \in \widehat{G}$ and $\hat{x} \neq \hat{x}_k$, and $F(\tau(\hat{x}_k), \nu(\hat{x}_k)) \leq Z_0$ for $\hat{x} = \hat{x}_k$. Then obviously $(\tau, \nu) \in K_h$, and so

$$(\dot{\epsilon}(\hat{x}_k) - C\dot{\sigma}(\hat{x}_k), \tau(\hat{x}_k) - \sigma(\hat{x}_k)) + (-\dot{\alpha}(\hat{x}_k), \nu(\hat{x}_k) - \alpha(\hat{x}_k)) \leq 0 \quad . \tag{7.10}$$

If $(\hat{x}_k, t)$ satisfies $F(\sigma(\hat{x}_k, t), \alpha(\hat{x}_k, t)) < Z_0$, by the continuity of the gauge function $F(\sigma, \alpha)$, we can conclude from (7.10) that

$$\begin{cases} \dot{\epsilon}(\hat{x}_k, t) = C\dot{\sigma}(\hat{x}_k, t) \\ \dot{\alpha}(\hat{x}_k, t) = 0 \end{cases} \quad \text{or} \quad \begin{cases} \dot{\sigma}(\hat{x}_k, t) = D\dot{\epsilon}(\hat{x}_k, t) \\ \dot{\alpha}(\hat{x}_k, t) = 0 \end{cases} \tag{7.11}$$

which is just (4.2).

If $(\hat{x}_k, t)$ satisfies $F(\sigma(\hat{x}_k, t), \alpha(\hat{x}_k, t)) = Z_0$, then from the uniform boundedness, $|\dot{\sigma}(\hat{x}_k)| \leq C\|\dot{f}\|_{L^2(\Omega)}$ and $|\dot{\alpha}(\hat{x}_k)| \leq C\|\dot{f}\|_{L^2(\Omega)}$, we know that σ and α are absolutely continuous with respect to time t. So if we define $g(t) = F(\sigma(\hat{x}_k, t), \alpha(\hat{x}_k, t))$, because $F(\sigma, \alpha)$ is analytic with respect to σ and α, $g(t)$ is differentiable *a.e.* on $[0, T]$. Hence

$$\lim_{h \to 0} \frac{g(t+h) - g(t)}{h} \leq 0 \qquad \text{because} \qquad g(t+h) \leq Z_0 = g(t)$$

$$\lim_{h \to 0} \frac{g(t) - g(t-h)}{h} \geq 0 \qquad \text{because} \qquad g(t-h) \leq Z_0 = g(t)$$

and so we get $\frac{d}{dt}g(t) = 0$, which means

$$\partial_\sigma F(\sigma(\hat{x}_k, t), \alpha(\hat{x}_k, t))^T \dot{\sigma}(\hat{x}_k; t) + \partial_\alpha F(\sigma(\hat{x}_k, t), \alpha(\hat{x}_k, t))^T \dot{\alpha}(\hat{x}_k, t) = 0 \quad . \tag{7.12}$$

Inequality (7.10) implies that there exists $\lambda(\hat{x}_k, t) \geq 0$ satisfying

$$\begin{bmatrix} \dot{\epsilon}(\hat{x}_k, t) - C\dot{\sigma}(\hat{x}_k, t) \\ -\dot{\alpha}(\hat{x}_k, t) \end{bmatrix} = \begin{bmatrix} \dot{\epsilon}^p(\hat{x}_k, t) \\ -\dot{\alpha}(\hat{x}_k, t) \end{bmatrix} = \lambda \begin{bmatrix} \partial_\sigma F(\sigma(\hat{x}_k, t), \alpha(\hat{x}_k, t)) \\ \partial_\alpha F(\sigma(\hat{x}_k, t), \alpha(\hat{x}_k, t)) \end{bmatrix} .$$

From (7.12) we get

$$\lambda(\hat{x}_k, t) = \frac{\partial_\sigma F(\sigma(\hat{x}_k, t), \alpha(\hat{x}_k, t))^T \dot{\sigma}(\hat{x}_k, t)}{\partial_\alpha F(\sigma(\hat{x}_k, t), \alpha(\hat{x}_k, t))^T \partial_\alpha F(\sigma(\hat{x}_k, t), \alpha(\hat{x}_k, t))} .$$

So if $\lambda(\hat{x}_k, t) > 0$, we have

$$\partial_\sigma F(\sigma(\hat{x}_k, t), \alpha(\hat{x}_k, t))^T \dot{\sigma}(\hat{x}_k, t) > 0 \quad .$$

Hence

$$\begin{cases} \dot{\epsilon}(\hat{x}_k, t) - C\dot{\sigma}(\hat{x}_k, t) = \dfrac{\partial_\sigma F(\sigma, \alpha)^T \dot{\sigma}}{\partial_\alpha F(\sigma, \alpha)^T \partial_\alpha F(\sigma, \alpha)} \partial_\sigma F(\sigma, \alpha)(\hat{x}_k, t) \\ \dot{\alpha}(\hat{x}_k, t) = -\dfrac{\partial_\sigma F(\sigma, \alpha)^T \dot{\sigma}}{\partial_\alpha F(\sigma, \alpha)^T \partial_\alpha F(\sigma, \alpha)} \partial_\alpha F(\sigma, \alpha)(\hat{x}_k, t) \end{cases}$$

Solving for $\dot\sigma(\hat x_k, t)$ in the first equation, we get

$$\begin{cases} \dot\sigma(\hat x_k, t) = \left(D - \dfrac{D\partial_\sigma F \partial_\sigma F^T D}{\partial_\alpha F^T \partial_\alpha F + \partial_\sigma F^T D \partial_\sigma F}\right)(\hat x_k, t)\,\dot\epsilon(\hat x_k, t) \\ \dot\alpha(\hat x_k, t) = -\dfrac{\partial_\sigma F(\hat x_k, t)^T \dot\sigma(\hat x_k, t)}{(\partial_\alpha F^T \partial_\alpha F)(\hat x_k, t)}\,\partial_\alpha F(\hat x_k, t) \end{cases}$$

which is just nothing else but (4.3).

If $\lambda(\hat x_k, t) = 0$, then we have

$$\partial_\sigma F(\sigma(\hat x_k, t), \alpha(\hat x_k, t))^T \dot\sigma(\hat x_k, t) = 0 \quad .$$

In this case, we again get (4.2) and the proof is completed. □

8 Convergence Analysis of the Semi-discrete Problem

Let $(U_h, \sigma_h, \alpha_h)$ be the unique solution of the semi-discrete problem (7.2) ~ (7.5) associated with the partition $\mathcal{T}_h$ of the domain Ω. Then we have $(U_h, \sigma_h, \alpha_h)$ satisfying:

$$\begin{cases} \int_\Omega \left(\dot\sigma_{h_{11}} \dfrac{\partial \phi_n}{\partial x_1} + \dot\sigma_{h_{12}} \dfrac{\partial \phi_n}{\partial x_2}\right) dx = \int_\Omega (\dot f_1, \phi_n)\, dx \\ \int_\Omega \left(\dot\sigma_{h_{21}} \dfrac{\partial \phi_n}{\partial x_1} + \dot\sigma_{h_{22}} \dfrac{\partial \phi_n}{\partial x_2}\right) dx = \int_\Omega (\dot f_2, \phi_n)\, dx \end{cases} \quad n \in \mathcal{N} \tag{8.1}$$

$$\int_{\Omega\times I} [(\dot\epsilon_h - C\dot\sigma_h, \tau - \sigma_h) + (-\dot\alpha_h, \nu - \alpha_h)]\, dx\, dt \le 0 \quad \forall (\tau, \nu) \in K_h \tag{8.2}$$

with

$$\dot U_h \in (Q_{1,0}(\Omega_h))^2 \subset (H_0^1(\Omega))^2 \qquad \text{for all} \quad t \in I \tag{8.3}$$

and the initial conditions

$$\begin{cases} U_h(0, x) = 0 \\ \sigma_h(0, x) = 0 \\ \alpha_h(0, x) = 0 \end{cases} \tag{8.4}$$

From Theorem 6.1, we know that

$$\left.\begin{array}{l} \nabla U_h, U_h, \epsilon_h, \alpha_h \\ \\ \nabla \dot U_h, \dot U_h, \dot\epsilon_h, \dot\alpha_h \end{array}\right\} \quad \begin{array}{l} \text{are uniformly bounded} \\ \\ \text{in } L^\infty(I, L^2(\Omega)) \quad . \end{array}$$

Therefore we can extract a subsequence (See, e.g. [15] & [16]), such that

$$\nabla U_h, U_h, \sigma_h, \alpha_h, \epsilon_h \xrightarrow{\;*\;} \nabla U, U, \sigma, \alpha, \epsilon$$

$$\nabla \dot U_h, \dot U_h, \dot\sigma_h, \dot\alpha_h, \dot\epsilon_h \xrightarrow{\;*\;} \nabla \dot U, \dot U, \dot\sigma, \dot\alpha, \dot\epsilon$$

weakly* in $L^\infty(I, L^2(\Omega))$. For the limit functions σ, α, ϵ and U, we have the following theorem.

Theorem 8.1 *The limit functions* (U,σ,α) *is the unique solution of the following problem*

$$\begin{cases} \int_\Omega (\dot\sigma_{11}\frac{\partial\phi}{\partial x_1} + \dot\sigma_{12}\frac{\partial\phi}{\partial x_2})\,dx = \int_\Omega (\dot f_1,\phi)\,dx \\ \int_\Omega (\dot\sigma_{21}\frac{\partial\phi}{\partial x_1} + \dot\sigma_{22}\frac{\partial\phi}{\partial x_2})\,dx = \int_\Omega (\dot f_2,\phi)\,dx \end{cases} \quad \forall\,\phi \in C_0^\infty(\Omega) \tag{8.5}$$

a.e. $t \in I$ with $(\sigma,\alpha) \in K$, where

$$K = \Big\{ (\sigma,\alpha) \in L^\infty(I, L^2(\Omega)) \,|\, F(\sigma,\alpha)(x,t) \le Z_0 \quad a.e.\ \ \Omega\times I \Big\} \ , \tag{8.6}$$

and the admissible equation

$$\int_{\Omega\times I} \Big[(\dot\epsilon - C\dot\sigma, \tau-\sigma) + (-\dot\alpha, \nu-\alpha) \Big]\,dx\,dt \le 0 \qquad \forall\,(\tau,\nu) \in K \tag{8.7}$$

with

$$U,\dot U \in L^\infty(I, H_0^1(\Omega)) \ , \quad \sigma,\dot\sigma,\alpha,\dot\alpha \in L^\infty(I, L^2(\Omega)) \ ,$$

and the initial conditions

$$\begin{cases} U(x,0) = 0 \\ \sigma(x,0) = 0 \\ \alpha(x,0) = 0 \end{cases} \qquad a.e.\ \text{on}\ \Omega \tag{8.8}$$

PROOF: We will prove this theorem in five steps.

Step 1. To show that the limit functions (U,σ,α) also satisfy the initial condition (8.8). When choosing the subsequence from (U_h,σ_h,α_h) we can also assume that

$$U_h,\dot U_h \xrightarrow{\ *\ } U,\dot U \quad \text{weakly}^* \ \text{in}\ L^\infty(I, H_0^1(\Omega)) \ , \tag{8.9}$$

and
$$U_h \longrightarrow U \quad \text{weakly in}\ H^1(\Omega\times I),$$

which implies that $U_h, U \in AC(I, L^2(\Omega))$, (See, e.g. [15] Chapter 11). So for any $\phi(x) \in L^2(\Omega)$, we can take $\psi(x,t) = \frac{t-T}{T}\phi(x)$; and get

$$\psi(x,t) \in L^2(I, L^2(\Omega)) \ ,$$

$$\frac{\partial\psi(x,t)}{\partial t} \in L^2(I, L^2(\Omega)) \ .$$

Since $U(x,t) \in AC(I, L^2(\Omega))$, $U(x,0)$ is well defined and

$$U(x,0)\phi(x) = -\int_0^T U(x,t)\frac{\partial\psi(x,t)}{\partial t}\,dt - \int_0^T \dot U(x,t)\,\psi(x,t)\,dt$$

a.e. x in Ω. So we get

$$\int_\Omega U(x,0)\phi(x)\,dx = -\int_{\Omega\times I} U(x,t)\frac{\partial\psi(x,t)}{\partial t} - \int_{\Omega\times I} \dot U(x,t)\,\psi(x,t) \ .$$

Similarly for the function $U_h(x,t)$, we have

$$\int_\Omega U_h(x,0)\phi(x)dx = -\int_{\Omega\times I} U_h(x,t)\frac{\partial\psi(x,t)}{\partial t} - \int_{\Omega\times I} \dot{U}_h(x,t)\psi(x,t) \quad .$$

Thus, by (8.9), we get

$$\lim_{h\to 0}\int_\Omega U_h(x,0)\phi(x)dx = \int_\Omega U(x,0)\phi(x)dx \qquad \forall\,\phi(x)\in L^2(\Omega) \quad ,$$

which means that the subsequence $U_h(x,0) \longrightarrow U(x,0)$ weakly in $L^2(\Omega)$, and so we have $U(x,0)=0$ *a.e.* on Ω. Similarly we can show that $\sigma(x,0)=0$, $\alpha(x,0)=0$ *a.e.* on Ω, and all the initial conditions are satisfied.

Step 2. To show the system (8.5) holds. Indeed, for any piecewise bilinear function $\phi_h \in Q_{1,0}(\Omega_h)$, we have

$$\begin{cases} \int_\Omega (\dot\sigma_{h_{11}}\frac{\partial\phi_h}{\partial x_1} + \dot\sigma_{h_{12}}\frac{\partial\phi_h}{\partial x_2})\,dx = \int_\Omega (\dot f_1,\phi_h)\,dx \\ \int_\Omega (\dot\sigma_{h_{21}}\frac{\partial\phi_h}{\partial x_1} + \dot\sigma_{h_{22}}\frac{\partial\phi_h}{\partial x_2})\,dx = \int_\Omega (\dot f_2,\phi_h)\,dx \end{cases} \quad \forall\,\phi_h\in Q_1(\Omega_h) \tag{8.10}$$

a.e. $t\in I$. So for any $\phi(x,t)\in C_0^\infty(\Omega)\times C^\infty(I)$, we can find a sequence of simple abstract functions $\hat\phi_h(x,t)\in L^\infty(I,Q_{1,0}(\Omega_h)\cap H_0^1(\Omega))$ such that $\hat\phi_h(x,t)$ converges to $\phi(x,t)$ strongly in $H^1(\Omega\times I)$, *i.e.*

$$\lim_{h\to 0}\int_{\Omega\times I}(\hat\phi_h-\phi)^2 dx\,dt = 0 \tag{8.11}$$

and

$$\lim_{h\to 0}\int_{\Omega\times I}(\frac{\partial\hat\phi_h}{\partial x_i}-\frac{\partial\phi}{\partial x_i})^2 dx\,dt = 0 \qquad i=1,2 \quad . \tag{8.12}$$

For any $i,j,k=1,2$, we have

$$\begin{aligned} &\int_{\Omega\times I}\dot\sigma_{h_{ij}}\frac{\partial\hat\phi_h}{\partial x_k}dx\,dt - \int_{\Omega\times I}\dot\sigma_{ij}\frac{\partial\phi}{\partial x_k}dx\,dt \\ &= \int_{\Omega\times I}(\dot\sigma_{h_{ij}}-\dot\sigma_{ij})\frac{\partial\phi}{\partial x_k}dx\,dt + \int_{\Omega\times I}\dot\sigma_{h_{ij}}\left(\frac{\partial\hat\phi_h}{\partial x_k}-\frac{\partial\phi}{\partial x_k}x_k\right)dx\,dt \quad . \end{aligned} \tag{8.13}$$

From the weak* convergence of the sequence σ_h, the first term of the right hand side of (8.13) will tend to zero as $h\longrightarrow 0$. For the second term of the right hand side of (8.13) we have the following inequality:

$$\begin{aligned} \left|\int_{\Omega\times I}\dot\sigma_{h_{ij}}\left(\frac{\partial\hat\phi_h}{\partial x_k}-\frac{\partial\phi}{\partial x_k}\right)dx\,dt\right| &\le \|\dot\sigma_{h_{ij}}\|_{L^2(\Omega\times I)}\left\|\frac{\partial\hat\phi_h}{\partial x_k}-\frac{\partial\phi}{\partial x_k}\right\|_{L^2(\Omega\times I)} \\ &\le \|\dot\sigma_{h_{ij}}\|_{L^2(\Omega\times I)}\left\|\hat\phi_h-\phi\right\|_{H^1(\Omega\times I)} \quad . \end{aligned}$$

From energy estimates, we know that $\|\dot\sigma_{h_{ij}}\|_{L^2(\Omega\times I)}$ is uniformly bounded. Meanwhile, from the strong H^1 convergence of $\hat\phi_h$ to ϕ, we can see that the second term of the right hand side of (8.13) will also tend to zero. Obviously

$$\lim_{h\to 0}\int_{\Omega\times I}(\dot f_i,\hat\phi_h)\,dx\,dt = \int_{\Omega\times I}(\dot f_i,\phi)\,dx\,dt \qquad i=1,2 \quad .$$

So as $h \longrightarrow 0$, we can get, from (8.10), that

$$\begin{cases} \int_{\Omega\times I} (\dot\sigma_{11} \frac{\partial \phi}{\partial x_1} + \dot\sigma_{12} \frac{\partial \phi}{\partial x_2}) \, dx\, dt = \int_{\Omega\times I} (\dot f_1, \phi)\, dx \\ \int_{\Omega\times I} (\dot\sigma_{21} \frac{\partial \phi}{\partial x_1} + \dot\sigma_{22} \frac{\partial \phi}{\partial x_2}) \, dx\, dt = \int_{\Omega\times I} (\dot f_2, \phi)\, dx \end{cases} \quad \forall \ \phi \in C_0^\infty(\Omega) \times C^\infty(I)$$

which means (8.5) holds.

Step 3. To show that the limit function (U, σ, α) still satisfies (8.7). Let $(\tau, \nu) \in K$, then we can claim that there exists a sequence $(\tau_h, \nu_h) \in K_h$ such that

$$\begin{cases} \tau_h \longrightarrow \tau \\ \nu_h \longrightarrow \nu \end{cases} \qquad \text{strongly in } L^2(\Omega \times I) \ .$$

Indeed, actually we can construct such a sequence as follows: consider a sequence of mollifiers J_ϵ defined on $\mathbb{R}^2 \times \mathbb{R}$ (See [1] for details). Then we have

$$\max_{x \in \mathbb{R}^2\times\mathbb{R}} J_\epsilon(x) = J_\epsilon(0) = \frac{k}{\epsilon^3}$$

and

$$\int_{\mathbb{R}^2\times\mathbb{R}} J_\epsilon(x)\, dx = 1 \ .$$

Let $\hat\Omega$, $\hat I$ be the two sets such that $\Omega \subset\subset \hat\Omega$, $I \subset\subset \hat I$ and $\operatorname{dist}(\Omega, \partial\hat\Omega) > 1$, $\operatorname{dist}(I, \partial\hat I) > 1$. Define $\tau(x) = \nu(x) = 0$ for all $x \in (\mathbb{R}^2 \times \mathbb{R}) \backslash (\Omega \times I)$. For any $z \in \Omega \times I$ and $\epsilon < 1$, take

$$\tau_\epsilon(z) = \int_{\mathbb{R}^2\times\mathbb{R}} J_\epsilon(y)\tau(z-y)\, dy = \int_{\hat\Omega\times\hat I} J_\epsilon(y)\tau(z-y)\, dy \ ,$$

$$\nu_\epsilon(z) = \int_{\mathbb{R}^2\times\mathbb{R}} J_\epsilon(y)\nu(z-y)\, dy = \int_{\hat\Omega\times\hat I} J_\epsilon(y)\nu(z-y)\, dy \ .$$

Then by Lemma 2.18 of [1], we have $\tau_\epsilon(z), \nu_\epsilon(z) \in C_0^\infty(\hat\Omega \times \hat I)$ when ϵ is small enough, and $\tau_\epsilon \longrightarrow \tau$, $\nu_\epsilon \longrightarrow \nu$ strongly in $L^2(\Omega \times I)$ as $\epsilon \longrightarrow 0$. On the other hand, from the Jenssen's Inequality (See, for example [7] or [16]), for any convex function $G(u)$ and any integrable function $u(x,t)$ over $\Omega \times I$, we have

$$G\Big(\frac{1}{m(\Omega\times I)} \int_{\Omega\times I} u(x,t)\, dx\, dt\Big) \le \frac{1}{m(\Omega\times I)} \int_{\Omega\times I} G(u(x,t))\, dx\, dt \tag{8.14}$$

or equivalently

$$G\Big(\int_{\Omega\times I} u(x,t)\, dx\, dt\Big) \le \frac{1}{C\, m(\Omega\times I)} \int_{\Omega\times I} G(C\, m(\Omega \times I)\, u(x,t))\, dx\, dt \ , \tag{8.15}$$

where $m(\Omega \times I)$ is the measure of the set $\Omega \times I$ and C is an arbitrary constant. So for our gauge function $F(\sigma, \alpha)$ and functions $\tau_\epsilon(z)$ and $\nu_\epsilon(z)$, we have

$$F(\tau_\epsilon(z), \nu_\epsilon(z)) = F\Big(\int_{\hat\Omega\times\hat I} J_\epsilon(y)\tau(z-y)\, dy, \int_{\hat\Omega\times\hat I} J_\epsilon(y)\nu(z-y)\, dy\Big)$$

$$\le \frac{1}{C_\epsilon m(\hat\Omega\times\hat I)} \int_{\hat\Omega\times\hat I} F(C_\epsilon m(\hat\Omega\times\hat I) J_\epsilon(y)\tau(z-y), C_\epsilon m(\hat\Omega\times\hat I) J_\epsilon(y)\nu(z-y))\, dy \ .$$

If we choose $C_\epsilon = \dfrac{\epsilon^3}{k\,m(\hat{\Omega}\times\hat{I})}$, we would have $C_\epsilon\, m(\hat{\Omega}\times\hat{I})\, J_\epsilon(y) \le 1 \quad \forall \quad y \in \hat{\Omega}\times\hat{I}$. So from the convexity of the gauge function $F(\tau,\nu)$ and the assumption: $F(0,0) = 0$, we get

$$F(\lambda_\epsilon \tau(x-y), \lambda_\epsilon \nu(x-y))$$

$$= F(\lambda_\epsilon \tau(x-y) + (1-\lambda_\epsilon)0, \lambda_\epsilon \nu(x-y) + (1-\lambda_\epsilon)0)$$

$$\le \lambda_\epsilon F(\tau(x-y), \nu(x-y)) + (1-\lambda_\epsilon) F(0,0) = \lambda_\epsilon F(\tau(x-y), \nu(x-y)) \quad ,$$

where $0 \le \lambda_\epsilon = C_\epsilon\, m(\hat{\Omega}\times\hat{I})\, J_\epsilon(y) \le 1$. Hence we get

$$F(\tau_\epsilon(x), \nu_\epsilon(x)) \le \int_{\hat{\Omega}\times\hat{I}} J_\epsilon(y) F(\tau(x-y), \nu(x-y))\, dy \le Z_0 \int_{\mathbf{R}^3\times\mathbf{R}} J_\epsilon(y)\, dy = Z_0 \quad ,$$

which implies $(\tau_\epsilon, \nu_\epsilon) \in K$. Because $(\tau_\epsilon(y), \nu_\epsilon(y)) \in C^\infty(\Omega\times I)$, for any $t \in I$ we can interpolate $(\tau_\epsilon(y), \nu_\epsilon(y))$ at the Gaussian points by the basis function $(\tilde{\phi}^h_n)$ on T_h and construct a sequence of simple abstract functions (τ_h, ν_h) such that

$$(\tau_h, \nu_h) \in K_h \qquad\qquad \forall \quad t \in I$$

and

$$(\tau_h, \nu_h) \longrightarrow (\tau, \nu) \quad \text{strongly in } L^2(\Omega\times I) \quad .$$

Taking $(\tau, \nu) = (\tau_h, \nu_h)$ in (8.2), we get

$$\int_{\Omega\times I} (\dot{\epsilon}_h - C\dot{\sigma}_h, \tau_h - \sigma_h)\, dx\, dt - (\dot{\alpha}_h, \nu_h - \alpha_h)\, dx\, dt \le 0 \quad ,$$

which can be written in the form

$$\int_{\Omega\times I} [(\dot{\epsilon}_h - C\dot{\sigma}_h, \tau_h) - (\dot{\alpha}_h, \nu_h)]\, dx\, dt - \int_{\Omega\times I} (\sigma_h, \dot{\epsilon}_h)\, dx\, dt$$
$$+ \frac{1}{2}\int_{\Omega\times I} \left[\frac{d}{dt}(C\sigma_h, \sigma_h) + \frac{d}{dt}(\alpha_h, \alpha_h)\right] dx\, dt \le 0 \quad .$$

The weak* convergence of the sequence $(U_h, \sigma_h, \alpha_h)$ and the strong convergence of the sequence (τ_h, ν_h) will ensure that the first term tends as $h \longrightarrow 0$ to

$$\int_{\Omega\times I} [(\dot{\epsilon} - C\dot{\sigma}) - (\dot{\alpha}, \nu)]\, dx\, dt \quad ;$$

the second term is equal to

$$\int_{\Omega\times I} (f, \dot{U}_h)\, dx\, dt, \qquad \text{and hence tends to} \qquad \int_{\Omega\times I} (f, \dot{U})\, dx\, dt \quad .$$

Because the initial conditions: $\sigma(x,0) = 0$, $\alpha(x,0) = 0$, the third term is equal to

$$\frac{1}{2}\left[\left\|\sqrt{C}\,\sigma_h(T)\right\|^2_{L^2(\Omega)} + \|\alpha_h(T)\|^2_{L^2(\Omega)}\right] \quad ,$$

and

$$\liminf_{h\longrightarrow 0} \left\|\sqrt{C}\,\sigma_h(T)\right\|^2_{L^2(\Omega)} \ge \left\|\sqrt{C}\,\sigma(T)\right\|^2_{L^2(\Omega)} \quad ,$$

$$\liminf_{h\longrightarrow 0} \|\alpha_h(T)\|^2_{L^2(\Omega)} \ge \|\alpha(T)\|^2_{L^2(\Omega)} \quad .$$

So we end up as $h \longrightarrow 0$ with

$$\int_{\Omega\times I}[(\dot{\epsilon} - C\dot{\sigma},\tau) - (\dot{\alpha},\nu)] - \int_{\Omega\times I}(f,\dot{U}) + \frac{1}{2}\left[\left\|\sqrt{C}\sigma(T)\right\|^2_{L^2(\Omega)} + \|\alpha(T)\|^2_{L^2(\Omega)}\right] \le 0 \quad .$$

From (8.5) and the fact that

$$\frac{1}{2}\|\alpha(T)\|^2_{L^2(\Omega)} = \int_{\Omega\times I}(\dot{\alpha},\alpha)\,dx\,dt \quad ,$$

and

$$\frac{1}{2}\left\|\sqrt{C}\,\sigma(T)\right\|^2_{L^2(\Omega)} = \int_{\Omega\times I}(C\dot{\sigma},\sigma)\,dx\,dt \quad ,$$

we can finally get (8.7).

Step 4. To show that $(\sigma,\alpha) \in K$, i.e. $F(\sigma,\alpha) \le Z_0$ *a.e.* on $\Omega \times I$. Let $P_{\hat{K}} : \mathbb{R}^3 \times \mathbb{R}^m \longmapsto \mathbb{R}^3 \times \mathbb{R}^m$ be the projection operator on the convex set

$$\hat{K} = \left\{(\tau,\nu) \in \mathbb{R}^3 \times \mathbb{R}^m \;\middle|\; F(\tau,\nu) \le Z_0\right\} \quad .$$

For the convex set $\hat{K}$, we can use the projection theorem and get that for any $(\sigma,\alpha) \in \mathbb{R}^3 \times \mathbb{R}^m$ and any $(\tau,\nu) \in \hat{K}$,

$$\left[\begin{bmatrix}\sigma\\ \alpha\end{bmatrix} - P_{\hat{K}}\begin{bmatrix}\sigma\\ \alpha\end{bmatrix}, \begin{bmatrix}\tau\\ \nu\end{bmatrix} - P_{\hat{K}}\begin{bmatrix}\sigma\\ \alpha\end{bmatrix}\right] \le 0 \quad .$$

Therefore we get

$$\int_{\Omega\times I}\left[\begin{bmatrix}\sigma\\ \alpha\end{bmatrix} - P_{\hat{K}}\begin{bmatrix}\sigma\\ \alpha\end{bmatrix}, \begin{bmatrix}\tau\\ \nu\end{bmatrix} - P_{\hat{K}}\begin{bmatrix}\sigma\\ \alpha\end{bmatrix}\right]dx\,dt \le 0 \quad \forall(\tau,\nu) \in \hat{K} \quad .$$

For any $t \in I$, let $\hat{\sigma}_h$ and $\hat{\alpha}_h$ be the piecewise constant functions on Ω such that on each rectangular element T_n their values are defined by:

$$\hat{\sigma}_h := \sum_{k=1}^{4}\frac{\omega_k}{S_n}\sigma_h(\hat{x}_k) \quad , \quad \hat{\alpha}_h := \sum_{k=1}^{4}\frac{\omega_k}{S_n}\alpha_h(\hat{x}_k) \quad , \tag{8.16}$$

where S_n is the area of the rectangular element T_n. Then from the fact that the gauge function $F(\sigma,\alpha)$ is convex and

$$\sum_{k=1}^{4}\frac{\omega_k}{S_n} = 1 \quad ,$$

we have

$$F(\hat{\sigma}_h,\hat{\alpha}_h) \le \sum_{k=1}^{4}\frac{\omega_k}{S_n}F(\sigma_h(\hat{x}_k),\alpha_h(\hat{x}_k)) \le \sum_{k=1}^{4}\frac{\omega_k}{S_n}Z_0 = Z_0 \quad ,$$

i.e.

$$(\hat{\sigma}_h,\hat{\alpha}_h) \in \hat{K} \qquad a.e. \text{ on } \Omega \times I \quad .$$

Consequently we have

$$\left[\begin{bmatrix}\sigma\\ \alpha\end{bmatrix} - P_{\hat{K}}\begin{bmatrix}\sigma\\ \alpha\end{bmatrix}, \begin{bmatrix}\hat{\sigma}_h\\ \hat{\alpha}_h\end{bmatrix} - P_{\hat{K}}\begin{bmatrix}\sigma\\ \alpha\end{bmatrix}\right] \le 0 \quad a.e. \text{ on } \Omega \times I \quad . \tag{8.17}$$

For the function $\begin{bmatrix} \sigma \\ \alpha \end{bmatrix} - P_{\hat{K}} \begin{bmatrix} \sigma \\ \alpha \end{bmatrix}$, we can find a sequence of abstract functions $\begin{bmatrix} \tau_h \\ \beta_h \end{bmatrix}$ such that for any fixed $t \in I$ they are piecewise constant, and as $h \longrightarrow 0$:

$$\begin{bmatrix} \tau_h \\ \beta_h \end{bmatrix} \xrightarrow{L^2(\Omega \times I)} \begin{bmatrix} \sigma \\ \alpha \end{bmatrix} - P_{\hat{K}} \begin{bmatrix} \sigma \\ \alpha \end{bmatrix} \quad . \tag{8.18}$$

Now we can rewrite (8.17) and get

$$\begin{aligned} &\left[\begin{bmatrix} \tau_h \\ \beta_h \end{bmatrix}, \begin{bmatrix} \hat{\sigma}_h \\ \hat{\alpha}_h \end{bmatrix} - P_{\hat{K}} \begin{bmatrix} \sigma \\ \alpha \end{bmatrix}\right] + \\ &\quad \left[\begin{bmatrix} \sigma \\ \alpha \end{bmatrix} - P_{\hat{K}} \begin{bmatrix} \sigma \\ \alpha \end{bmatrix} - \begin{bmatrix} \tau_h \\ \beta_h \end{bmatrix}, \begin{bmatrix} \hat{\sigma}_h \\ \hat{\alpha}_h \end{bmatrix} - P_{\hat{K}} \begin{bmatrix} \sigma \\ \alpha \end{bmatrix}\right] \le 0 \quad . \end{aligned} \tag{8.19}$$

Integrating the first term of (8.19) over $\Omega \times I$,

$$\begin{aligned} &\int_{\Omega \times I} \left[\begin{bmatrix} \tau_h \\ \beta_h \end{bmatrix}, \begin{bmatrix} \hat{\sigma}_h \\ \hat{\alpha}_h \end{bmatrix} - P_{\hat{K}} \begin{bmatrix} \sigma \\ \alpha \end{bmatrix}\right] \\ &= \int_{\Omega \times I} \left[\begin{bmatrix} \tau_h \\ \beta_h \end{bmatrix}, \begin{bmatrix} \sigma_h \\ \alpha_h \end{bmatrix} - P_{\hat{K}} \begin{bmatrix} \sigma \\ \alpha \end{bmatrix}\right] + \int_{\Omega \times I} \left[\begin{bmatrix} \tau_h \\ \beta_h \end{bmatrix}, \begin{bmatrix} \hat{\sigma}_h \\ \hat{\alpha}_h \end{bmatrix} - \begin{bmatrix} \sigma_h \\ \alpha_h \end{bmatrix}\right] \quad . \end{aligned}$$

The second term of the right hand side of the above equation is equal to zero, because (τ_h, β_h) are piecewise constant and $(\hat{\sigma}_h, \hat{\alpha}_h)$ are defined by (8.16). So from (8.19) we get

$$\begin{aligned} &\int_{\Omega \times I} \left[\begin{bmatrix} \tau_h \\ \beta_h \end{bmatrix}, \begin{bmatrix} \sigma_h \\ \alpha_h \end{bmatrix} - P_{\hat{K}} \begin{bmatrix} \sigma \\ \alpha \end{bmatrix}\right] + \\ &\quad \int_{\Omega \times I} \left[\begin{bmatrix} \sigma \\ \alpha \end{bmatrix} - P_{\hat{K}} \begin{bmatrix} \sigma \\ \alpha \end{bmatrix} - \begin{bmatrix} \tau_h \\ \beta_h \end{bmatrix}, \begin{bmatrix} \hat{\sigma}_h \\ \hat{\alpha}_h \end{bmatrix} - P_{\hat{K}} \begin{bmatrix} \sigma \\ \alpha \end{bmatrix}\right] \le 0 \quad . \end{aligned}$$

From (8.18) we have, as $h \longrightarrow 0$,

$$\int_{\Omega \times I} \left[\begin{bmatrix} \sigma \\ \alpha \end{bmatrix} - P_{\hat{K}} \begin{bmatrix} \sigma \\ \alpha \end{bmatrix}, \begin{bmatrix} \sigma \\ \alpha \end{bmatrix} - P_{\hat{K}} \begin{bmatrix} \sigma \\ \alpha \end{bmatrix}\right] \le 0 \quad ,$$

i.e.

$$\begin{bmatrix} \sigma \\ \alpha \end{bmatrix} = P_{\hat{K}} \begin{bmatrix} \sigma \\ \alpha \end{bmatrix} \qquad a.e. \ \text{on} \ \ \Omega \times I \quad ,$$

and so $(\sigma, \alpha) \in K$, or $F(\sigma, \alpha) \le Z_0$ *a.e.* on $\Omega \times I$.

Step 5. The proof of the uniqueness follows exactly as the proof of the Theorem 7.1 and this completes the proof of the theorem. □

For the solution of (8.5)~(8.8), we have the following result:

Proposition 8.1 *The solution of the equation system (8.5)~ (8.8) is also the solution of the system (3.2) ~ (3.10).*

PROOF: Here we only need to show that (8.2)~(8.3) imply the existence of the scalar function $\lambda(x,t) \geq 0$ such that

$$\begin{cases} \dot{\epsilon} - C\dot{\sigma} = 0 \\ \dot{\alpha} = 0 \end{cases} \qquad a.e. \text{ on } \mathcal{E} \tag{8.20}$$

$$\begin{cases} \dot{\epsilon} - C\dot{\sigma} = \dot{\epsilon}^p = \lambda\,\partial_\sigma F \\ \dot{\alpha} = -\lambda\,\partial_\alpha F \end{cases} \qquad a.e. \text{ on } \mathcal{P} \tag{8.21}$$

where

$$\mathcal{E} = \Big\{ (x,t) \in \Omega \times I \;\mid\; F(\sigma,\alpha)(x,t) < Z_0 \quad \text{or} \quad \partial_\sigma F^T \dot{\sigma}(x,t) \leq 0 \Big\}$$

$$\mathcal{P} = \Big\{ (x,t) \in \Omega \times I \;\mid\; F(\sigma,\alpha)(x,t) = Z_0 \quad \text{and} \quad \partial_\sigma F^T \dot{\sigma}(x,t) > 0 \Big\}$$

Let $\delta > 0$ and denote by $Q_\delta(x_0,t_0)$ a cube of side length δ centered at (x_0,t_0). Then for any $(\tau,\nu) \in K$, we choose $(\hat{\tau},\hat{\nu})$ such that

$$\begin{cases} (\hat{\tau},\hat{\nu}) = (\sigma,\alpha) & \text{in} \quad (\Omega \times I)\backslash Q_\delta(x_0,t_0) \\ (\hat{\tau},\hat{\nu}) = (\tau,\nu) & \text{in} \quad Q_\delta(x_0,t_0) \end{cases}$$

Obviously we have $(\hat{\tau},\hat{\nu}) \in K$, and (8.3) yields

$$\int_{Q_\delta(x_0,t_0)} [(\dot{\epsilon} - C\dot{\sigma}, \tau - \sigma) + (-\dot{\alpha}, \nu - \alpha)]\,dx\,dt \leq 0 \quad \forall (\tau,\nu) \in K \quad .$$

Using the Lebesgue Differentiation Theorem (See, for example, [17]), we get as $\delta \longrightarrow 0$

$$(\dot{\epsilon} - C\dot{\sigma}, \tau - \sigma) + (-\dot{\alpha}, \nu - \alpha) \leq 0 \quad a.e. \text{ on } \Omega \times I \quad \forall (\tau,\nu) \in K \quad .$$

Then using the same approach as we used in the step 3 of the proof of the Theorem 7.1, we can show that there exists a $\lambda(x,t) \geq 0$ such that (8.20) and (8.21) hold. □

Remark 8.1 We can easily see that this method can be extended to the cases where the displacement and stress functions are approximated by piecewise higher order polynomial functions. For instance, if displacement function is approximated by piecewise Q_2 or Q_2' functions, we can approximate the corresponding stress function by piecewise Q_1 functions, and in this case, nine Gaussian points will be used for the constitutive equations.

Remark 8.2 Unfortunately this semi-discrete approximation method can only be directly applied to the cases where the partition of the domain Ω only contains rectangular and parallelogram elements. However, as we will see in the following section, for cases where the partition of the domain contains some other type of elements, similar approach can be used to develop some higher order methods.

9 Generalization to the Triangular Elements

It is desirable to find a method which can be applied to triangular elements and that also has the same properties as rectangular or parallelogram elements. Then we would be able to treat cases where Ω is any polygonal domain. In this section, we will briefly discuss a method of using piecewise higher order polynomials over triangular elements .

Consider the natural triangle of area A as shown in Figure 9.1, where the natural coordinates (α, β, γ) are:

$$\alpha = \frac{A_1}{A}, \quad \beta = \frac{A_2}{A}, \quad \gamma = \frac{A_3}{A} \quad ,$$

where A_1, A_2, A_3 and A are the area of each triangles. Hence we have:

$$0 = \alpha + \beta + \gamma - 1 \quad .$$

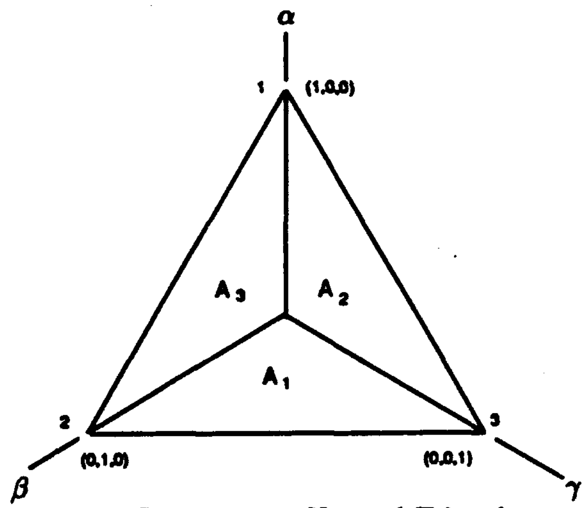

FIGURE 9.1 Natural Triangle and Coordinates

Now consider another standard triangle on the $X-Y$ plane as shown in Figure 9.2, and take any point (x, y) inside the triangle.

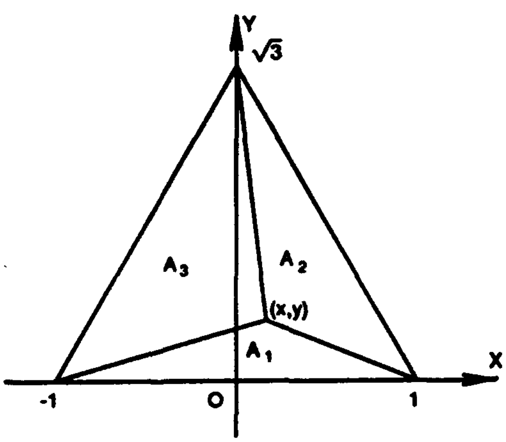

FIGURE 9.2 Standard Triangle on the X–Y Plane

Then a simple calculation shows:

$$A = \sqrt{3}$$

$$A_1 = y$$

$$A_2 = -\frac{1}{2}(\sqrt{3}x + y - \sqrt{3})$$

$$A_3 = \frac{1}{2}(\sqrt{3}x - y + \sqrt{3})$$

Therefore we get a one–to–one correspondence between the X–Y coordinate system and the natural coordinate system given by the following mapping functions:

$$\begin{cases} \alpha = \dfrac{1}{\sqrt{3}} y \\ \beta = -\dfrac{1}{2}(x + \dfrac{1}{\sqrt{3}} y - 1) \\ \gamma = \dfrac{1}{2}(x - \dfrac{1}{\sqrt{3}} y + 1) \end{cases} \qquad or \qquad \begin{cases} x = \gamma - \beta \\ y = \sqrt{3}\alpha \end{cases} \tag{9.1}$$

Therefore:

$$\int_{\Delta_{xy}} f(x,y)\,dx\,dy = \int_{\Delta_{\alpha\beta\gamma}} \tilde{f}(\alpha,\beta,\gamma)\,dA \quad ,$$

with $dA = 2A\,d\alpha\,d\beta$.

Some symmetrical Gaussian quadrature rules were discussed in [5], [6] and [9] , where an integration is performed by a Gaussian quadrature rule of the form:

$$\int_A f(\alpha,\beta,\gamma)\,dA = A\sum_{i=1}^{ng} \omega_i f(\alpha_i,\beta_i,\gamma_i) \quad . \tag{9.2}$$

In (9.2), $(\alpha_i,\beta_i,\gamma_i)$ are the natural coordinates of the ith Gaussian point, ω_i the corresponding Gaussian weight and ng is the number of Gaussian points used in the rule .

Table 9.1 Symmetrical Gaussian Quadrature Rules

p value	ng	weight	alpha	beta	gamma
1	1	1.0	1/3	1/3	1/3
2	3	1/3	2/3	1/6	1/6
4	6	0.223381590	0.108103018	0.445948491	0.445948491
		0.109951744	0.816847573	0.091576214	0.091576214
5	7	0.225000000	0.333333333	0.333333333	0.333333333
		0.132394152	0.059715872	0.470142064	0.470142064
		0.125939181	0.797426985	0.101286507	0.101286507
6	12	0.116786276	0.501426509	0.249286745	0.249286745
		0.050844906	0.873821971	0.063089014	0.063089014
		0.082851076	0.053145050	0.310352451	0.636502499
8	16	0.144315677	0.333333333	0.333333333	0.333333333
		0.095091634	0.081414823	0.459292588	0.459292588
		0.103217371	0.658861384	0.170569308	0.170569308
		0.032458498	0.898905543	0.050547228	0.050547228
		0.027230314	0.008394777	0.263112830	0.728492393
9	19	0.097135796	0.333333333	0.333333333	0.333333333
		0.031334700	0.020634962	0.489682520	0.489682520
		0.077827541	0.125820817	0.437089591	0.437089591
		0.079647739	0.623592929	0.188203536	0.188203536
		0.025577676	0.910540973	0.044729513	0.044729513
		0.043283539	0.036838412	0.221962989	0.741198599

A table of Gaussian quadrature rules of the form (9.2) for the polynomials of degree 1–20 was listed in [6]. Different from the Gaussian quadrature rules over rectangles, the Gaussian quadrature rules presented in [6] have some negative weight coefficients ω_i or some Gaussian points not lying inside the triangle A for some p values. In our case we cannot use Gaussian quadrature rules with negative weights, because, as we have seen in the previous discussion, the positiveness of the weight coefficients plays a very important role in the proofs of many theorems. We also cannot use those Gaussian quadrature rules with some Gaussian points lying outside the triangle, because we have to use extrapolation to determine the stress and hardening parameter functions. So in Table 9.1 we only list the rules with positive weights and all Gaussian points inside the triangle A .

In Table 9.1, for each weight if we have $\alpha = \beta = \gamma$, then there is only one Gaussian point corresponding to that weight coefficient. If we have $\alpha \neq \beta = \gamma$, then there are $C_3^2 = 3$ Gaussian points corresponding to that weight coefficient . Finally if we have $\alpha \neq \beta \neq \gamma$, then there are $C_3^1 = 6$ Gaussian points corresponding to that weight coefficient. Here we only list the weights and the location of the Gaussian points for the p values up to 9, for more data with $9 < p \leq 20$ see [6].

Now we can discuss the selection of the finite dimensional spaces for the displacement function $U(x,t)$ and the stress and hardening parameter functions $\sigma(x,t)$, $\alpha(x,t)$. The strategy is as follows: the finite dimensional spaces for U, σ and α should be selected in such a way that

i). The stress and hardening parameter functions σ, α will be uniquely determined by their values at the ng Gaussian points. (Assume here we want to use ng Gaussian Points)

ii). The degree of the polynomials of functions: σ^2, α^2 and $\sigma \cdot \epsilon(U)$ must be less than or equal to the p value listed in Table 9.1 corresponding to the number ng .

Table 9.2 gives a list of a possible selections of the spaces for U, σ and α. Here we always select the same spaces for σ and α.

Table 9.2 A Possible Selection of the Spaces for U, σ and α.

ng	D.O.P.	Space for U	Space for σ and α
1	1	P_1	P_0
3	2	P_2	P_1
6	4	P_3	P_2

For the complete polynomial subspaces, we can only get these three possible combinations. However, it is not necessary to always use complete polynomial subspaces for the displacement , stress and hardening functions. As a matter in fact, by using a Gaussian quadrature rule based on some non–complete polynomial subspaces, we can also get some higher order spaces for U, σ and α.

It is also not necessary to always choose piecewise polynomial functions as the subspace for the stress and hardening parameter functions. Since we don't require that σ, α are C^0 functions over the domain Ω. Actually we can select any n dimensional subspace $S_n \in L^2(\Omega)$ over each rectangle or triangle element, if this subspace has the following properties:

i). The stress and hardening parameter functions in this subspace will be uniquely determined by their values at some sampling points x_k $k = 1, \ldots, n$.

ii). There is a quadrature rule over the rectangle or triangle element with non–negative weights ω_k such that

$$\int f(x,y)\,dx\,dy = \sum_{k=1}^{n} \omega_k f(x_k, y_k)$$

for all the $f(x,y)$ of the forms:

$$(\epsilon(U(x,y)))^2, \quad \epsilon(U(x,y))\cdot g(x,y), \quad (g(x,y))^2$$

where $U(x,y)$ is any displacement function in its finite dimensional space and $g(x,y)$ is any function in the space S_n.

Finally we would like to indicate that the ideal of using Gaussian point may also be applied to the cases where the partition of the domain contains some curved elements. The convergence result we get here is only in the weak $L^2(\Omega \times I)$ sense, which is obviously not good enough for real computational purpose. However, we can prove that for some special constitutive laws, such as bilinear isotropic or bilinear kinematic hardening laws, we can actually get strong $L^2(\Omega \times I)$ convergence. For more general constitutive laws, the strong convergence is still achievable, if we use some extra assumptions on the the body traction functions.

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